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A REGIONAL ECONOMY, LAND USE, AND TRANSPORTATION MODEL (RELU-TRAN©): FORMULATION, ALGORITHM DESIGN, AND TESTING*

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ABSTRACT. RELU is a dynamic general equilibrium model of a metropolitan economy and its land use, derived by unifying in a theoretically valid way, models developed by one of the authors [Anas (1982), Anas–Arnott (1991, 1997), Anas–Kim (1996), Anas–Xu (1999)]. RELU equilibrates floor space, land and labor markets, and the market for the products of industries, treating development (construction and demolition), spatial interindustry linkages, commuting, and discretionary travel. Mode choices and equilibrium congestion on the highway network are treated by unifying RELU with the TRAN algorithm of stochastic user equilibrium [Anas–Kim (1990)]. The RELU-TRAN algorithm's performance for a stationary state is demonstrated for a prototype consisting of 4-building, 4-industry, 4-labor-type, 15-land-use-zone, 68-link-highway-network version of the Chicago MSA. The algorithm solves 656 equations in a special block-recursive convergent procedure by iterations nested within loops and loops within cycles. Runs show excellent and smooth convergence from different starting points, so that the number of loops within successive cycles continually decreases. The tests also imply a numerically ascertained unique stationary equilibrium solution of the unified model for the calibrated parameters.

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*Alex Anas dedicates this paper to the memory of Britton Harris (1915–2005), a visionary and devoted advocate for the development of better quantitative models for metropolitan planning. See Harris (1968, 1985). The research was funded by a 1998 National Science Foundation Urban Research Initiative award SES 9816816. Alex Anas was principal investigator and project director. The model developed and described here is currently being applied under a Science to Achieve Results (STAR) award from the Environmental Protection Agency (EPA). Yu Liu worked as research assistant to Alex Anas in the late stages of the NSF award. Unlike earlier assistants, he persevered and succeeded and is now post-doctoral fellow on the EPA award. The connections of RELU-TRAN to Anas' earlier papers co-authored with Liang-Shyong Duann, Richard Arnott, Ikki Kim, Rong Xu, and Hyok-Joo Rhee are evident and are hereby acknowledged. Any remaining deficiencies or errors belong to the authors. Aspects of this paper were presented at a seminar at the University of Illinois, Urbana (February 13, 2006), at the *First International Conference for Funding Transportation Infrastructure*, Banff, Canada (August 2–3, 2006), the *Double Workshop on Sustainable Cities and Transport*, Santiago, Chile (August 7–11, 2006), and the Keynote Address by Alex Anas at the 46th European Regional Science Association Congress in Volos, Greece, (August 31, 2006). The RELU, TRAN, and RELU-TRAN algorithms are copyrighted.

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Economic theory is rich in behavioral propositions which are useful in land use and transport modeling, and these fields have often unwisely neglected them. However, the converse is also true, and many of the pragmatic discoveries of modeling contained implications for economic behavior which were long neglected, or even worse denied. . . . planning and simulation have usually recognized that similar people in similar situations exhibit diverse behaviors. Gravity models capture that diversity, while standard economic models like linear programming do not. Only recently have the economists provided an alternative to their Procrustean bed in the form of discrete choice theory.

—Britton Harris (1985, p. 547).

1. INTRODUCTION

Not all economic models that are computationally challenging, interesting, and important conform to linear, quadratic, or other standard nonlinear programming formulations. Rather, such models require the solution of highly nonlinear equations systems using nonstandard and innovative, iterative algorithms that exploit the special features of those equations. This is the approach that is used to design the RELU-TRAN algorithm on which we report. Numerical solution of models using iterative techniques has been a goal, though poorly practiced within the field of transportation and land-use modeling. Meanwhile, iterative numerical methods are gaining broader applicability within economics to solve a variety of problems. See Judd (1998) for a survey and exposition of such methods and problems.

In this paper, the microeconomic structure of the RELU-TRAN model, and the algorithm designed to precisely solve its 656 equations for a stationary state are described and the algorithm's performance is documented. RELU-TRAN is a spatially disaggregated, easy to calibrate, computable general equilibrium model based on microeconomic theory and in which economic activity is modeled at the level of fully interdependent model zones with a link-node transport network, (see Figure 1). RELU-TRAN treats the stock of buildings in each model zone as changing slowly while other markets clear instantaneously. The metropolitan economy is treated as open in a number of ways. Consumers can, if they so choose, locate their residences or jobs outside the metropolitan area, income can originate from outside and a part of assets within the area can be owned by outsiders, while firms can produce, in part, by paying for inputs located elsewhere. The model treats interactions between firms and consumers and among firms as purely pecuniary, which are sufficient to generate a pattern of spatial agglomerations as in Anas and Xu (1999). Alternative sources of agglomeration such as those stemming from nonmarket interactions are not treated in the present version. It is well known that nonmarket externalities would generate multiple equilibria (e.g., Anas and Kim, 1996) and would thus require more sophisticated algorithms. The present model is nevertheless, suitable to study the effects of a menu of policies spanning capacity expansion, pricing, finance and investment of transportation, building and income taxation, and land-use planning and controls.

In order to see how RELU-TRAN extends prior developments, a brief overview is in order. Lowry's inspiring model (1964) was the first to recognize the importance of building a computable model of a metropolitan area. Lowry—limited by available data and no prior theory, since urban economics had barely emerged in 1964—had no choice but to use crude gravity models with ad hoc equilibration of land use. This modeling style was durably influential on a subsequent line of developments known as "Lowry-type models," exemplified by DRAM-EMPAL, lacking economic content (Putman, 1983), even after data and theory became available. Following Lowry, however, there were two important benchmark contributions by economists.

First, the NBER model of housing markets by Ingram, Kain, and Ginn (1972), made strides in computability, introduced better microeconomic content, and emphasized policy applicability. It inspired extensions that culminated in the Anas and Arnott model of housing markets (1991, 1997). Second, the general equilibrium model of metropolitan structure developed by Mills (1972) and extended by Hartwick and Hartwick (1974) and Kim (1979), utilized a linear programming based fixed-coefficient technology, but included a sophisticated treatment of traffic congestion on a grid geography with endogenous road capacities.[1] This linear programming structure, although fully consistent with economic theory and solvable using standard methods and the earlier partial equilibrium model of Herbert and Stevens (1960), have limitations that affect empirical relevance, computability and ease of calibration. These limitations are overcome by the use of discrete-choice models that are better suited to treating heterogeneity as explained by Anas (1982) and supported by Harris (1985). In the 1990s, Anas and Kim (1996), Anas and Xu (1999), and more recently Anas and Rhee (2007) extended the prior body of work to general equilibrium formulations by using discrete choice to model the joint choice of workplace, residence, and housing type while unifying it with the Dixit and Stiglitz (1977) representation of utility and production functions to generate budget-constrained discretionary trips for consumers and interindustry linkages for producers.

Meanwhile, the modeling of equilibrium traffic on congested highway networks by transportation scientists, has evolved largely separately from urban economics. Florian and Nguyen (1976) provided one of the earliest and Bar-Gera (2002) one of the most recent algorithms that find traffic equilibria on arbitrarily configured highway networks with fixed capacities, making operational the mathematical programming formulation of static traffic flow by Beckmann, McGuire, and Winsten (1958). However, being focused on transportation, these

[1]A dynamic version was later developed by Moore in his doctoral dissertation. See Moore and Wiggins (1990).

models take the land-use distribution and, often, the distribution of zone-to-zone trips as being fixed. This contrasts with urban economics where the interdependence of transportation and land use has been center stage from the beginnings of the field.

It has been recognized since the 1960s that a consistent operational metropolitan model, should integrate a model of congested traffic network equilibrium with a model of land use. This has led to a variety of efforts to devise *integrated transportation and land-use models* in both academia and planning practice without proper grounding in economics. In a first stream of such model integration, a rigorous, well-documented but partial form of model integration is sometimes attempted as an extension of travel demand analysis in order to improve travel forecasts. In such efforts, the origin-zone to destination-zone trip matrices are not fixed, as is common in conventional travel analysis, but respond in some cost-sensitive way to equilibrium travel costs on the networks. These "*integrated models of origin–destination, mode and route choice*" or "*combined models*" date back to Evans (1976). Only the trip generation, distribution, mode choice, and network equilibrium steps of travel forecasting are combined or treated with feedbacks but land use remains implicitly fixed. The state of the practice in these combined models is well documented in a recent presentation by Boyce and Florian (2005), who report clearly on large scale "*computational experiments for various integrated models of variable demand and auto route choice for a single, aggregated class of travelers plus trucks*." Although combined models are a step forward from the traditional four-step model of sequential travel forecasting, in none of these integration experiments—to our knowledge—are the variable travel demands made consistent with a land-use model, nor is a fully economic land-use or regional economy model made to respond to these travel demands. According to Boyce and Florian (2005), although the combined models used in planning practice show convergence to equilibrium in some large scale applications, they may not always converge well or to a high level of accuracy, but scholars (see, Bar-Gera and Boyce (2003), for example) are continuing to develop faster algorithms.

In a second stream of applications in metropolitan planning by consultants and practitioners, successful transportation and land-use model integration has been a goal. Models that have reportedly attempted this include MEPLAN, or the closely related TRANUS model of de la Barra (1989). These are not fully documented in scientific journals or in other publicly available forms, and cannot therefore be understood or evaluated completely. The MUSSA model of Martinez (1992, 1996), is better documented in scientific journals. The popular do-it-yourself modeling template of Wadell (1998), Urbansim, is open source and it is easy to verify that prices are not market clearing and that it does not conform to economic theory. Model integration in practice is highly demanding of technical personnel. Project deadlines, competition for funding, jitters about job security, and results urgently demanded by the users of the modeling work, as well as the lack of quality standards undoubtedly contribute to faulty model integration and to the opaque documentation of the results.[2] A goal of this paper is to establish a standard of documentation and transparency that befits scientific practice.

A summary of the rest of this paper is as follows. Section 2, supplemented by the notational glossary in the Appendix describes the microeconomic structure of the different sectors of RELU. These sectors are consumers, producers, landlords, and developers. Government is in the background via income tax rates that can be levied on consumers and ad valorem property tax rates on each type of building. Other taxes can be easily introduced. In the model, consumers make unified (simultaneously determined) choices of housing type, residential location, job location, and labor supply and choose a pattern of nonwork trips that originate from the residence location. These choices are consistent with utility maximization under time and money budgets as in Anas and Kim (1996) and Anas and Xu (1999) and are also hierarchically linked to the choice of travel mode (e.g., auto, public transit) for each trip and the choice of the minimum cost route on the congested highway network, nested within the travel mode choice.

Section 3 describes the general economic equilibrium of RELU, focusing on the excess demand equations for real estate markets, labor markets, the market for the product of each basic (export-oriented) industry, the markets for investment in business and real estate capital (buildings and land), and the composition of nonwage income. These discussions also clarify the senses in which the regional economy is open. To treat this openness, model zones are augmented by a number of peripheral zones. The peripheral zones can attract economic activity but are not equilibrated together with the model zones, and so rents, wages, and prices in the peripheral zones are treated as exogenous. Consumers can locate job or residence in a peripheral zone and make trips to a peripheral zone. Firms, locating in model zones utilize inputs that originate within the region as well as inputs that are located outside the region. Each model zone also functions as an export zone so that products from there can be directly shipped to the outside world. The income composition of the metropolitan economy includes nonwage income from assets that can be held in the rest of the world. Section 4 reports on the structure of the transportation model, TRAN, which is based on Anas and Kim's (1990) formulation of stochastic user equilibrium on highway networks.

Section 5 reports on the special block-recursive structure and the precise numerical testing of the unified RELU-TRAN equilibrium-finding algorithm with a 14 (plus one peripheral) zone version of the Chicago MSA that

[2]Difficulties encountered with a prior assistant in the development of RELU-TRAN are highly illustrative of these issues and have been documented in Anas (2005), a project document written as a report to the SUNY Research Foundation that may be made available upon request. The report shows the types of problems that arise when faulty model integration occurs and how such faulty model integration can be presented as not being faulty, slipping by others who are unwilling or unable to scrutinize the results.

corresponds to 656 equations. The algorithm consists of iterations nested within loops and loops nested within cycles. We report on various runs that show excellent and smooth convergence results so that the number of loops within successive cycles declines continually. Results also imply a numerically ascertained unique equilibrium solution of the unified model, although we do not have (nor have we attempted) a general uniqueness proof. Section 6 concludes and discusses how the unified RELU-TRAN model is a useful tool for policy analysis and describes ongoing plans for its application.

2. RELU MODEL STRUCTURE

Suppose that the metropolitan area is subdivided into $\Im$ model zones plus $\wp$ peripheral zones representing adjoining areas and the rest of the world as shown in Figure 1. Let $\Im' \equiv \Im + \wp$ denote all these zones. Each of the $\Im$ model zones is the potential site of the $k = 1, \ldots, \aleph_1$ housing types (e.g., single-family type and multiple-family type buildings) and the $k = 1, \ldots, \aleph_2$ business building types (e.g., commercial and industrial) and each such zone also contains some undeveloped land. There are $r = 1, \ldots, \Re$ interdependent basic industries that can produce in any of the zones and $f = 1, \ldots, F$ consumer types who can reside in any zone by renting housing there, can work in any zone and who buy retail goods in all the zones. For each building type, there are landlords who rent out the floor space and there is an industry (developers) that can construct or demolish floor space in each type of building. Given this setting, in this section we discuss the behavior of consumers, producers, landlords, and developers.[3]

Consumers

Consumers who are also potential workers are exogenously distributed among $f = 1, \ldots, F$ skill groups. N_f is the exogenous number of consumers in skill group f. "Skill" refers to exogenous ability in the labor market, but the wage and nonwage incomes for each f will be endogenous. Residences and workplaces can occur in any zone, $i = 1, \ldots, \Im'$. Each consumer makes a set of discrete and continuous choices. Without implying any sequence among them, the discrete choices are as follows. First, a consumer decides whether he will be employed or unemployed and in which zone to work if employed. We define a fictitious nonspatial "work zone" denoted by $j = 0$, to represent the voluntary choice of unemployment. Imagine that the choice of $j = 0$ entails zero wage income and is instantly accessible to every residence zone i. Thus, the consumer is depicted as choosing among the $j = 0, 1, \ldots, \Im'$ available zones (including peripheral) and this framework incorporates the choice of (voluntary) unemployment. Second, all consumers also choose a zone of residence among the $i = 1, \ldots, \Im'$ zones and,

[3]The notation nearly exhausts the English and Greek alphabets. Although symbols are defined in text, a Notational Glossary in which variables are systematically classified is presented in the Appendix for ease of reference.

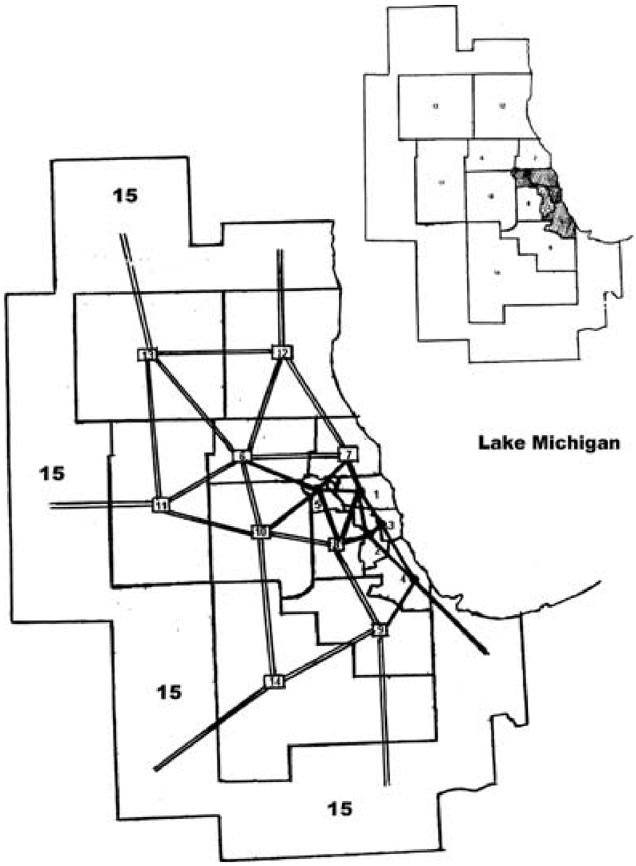

FIGURE 1: Model Region (Chicago MSA) and Aggregated Transport Network for Testing RELU-TRAN. (Zones 1–5 are the City of Chicago and Zones 6–14 Are the Suburbs. Zone 15 Is the Peripheral Area.)

third, they choose one of up to $k = 1, \ldots, \aleph_1$ housing types that are available in their zone of residence.

The remaining choices entail continuous variables and occur conditional on the discrete choices. Given their chosen work–residence–housing (i, j, k), consumers decide on: (i) the quantity of the chosen type-k housing floor space to rent; (ii) how many hours of labor to supply annually at the chosen workplace at some $j > 0$ (i.e., if employed); and (iii) what aggregate annual quantity of retail goods to purchase at each potential retail destination, z.[4] This mixed discrete-continuous utility maximization problem is:

[4]The discrete/continuous choice structure is Anas–Xu (1999), but they treated only raw land, and all consumers were employed and made a labor/leisure trade-off. In the benchmark RELU described here, labor supply is the residual time after the allocation of a time budget to work and nonwork trips, while leisure is implicitly fixed. Consequently, an unemployed traveler has no

(1)

$$\underset{\forall(i,j,k)}{\text{Max}}\left\{U^*_{ijk|f} \equiv \left[\begin{array}{l}\underset{\forall Z_z, b}{\text{Max}}\, U_{ijk|f} = \alpha_f \ln\left(\sum_{\forall z} \iota_{z|ijf}(Z_z)^{\eta_f}\right)^{\frac{1}{\eta_f}} + \beta_f \ln b + E_{ijk|f} + e_{ijk|f} \\ \text{subject to the budget}: \sum_{\forall z}(p_{\Re z} + c_{ijf} g_{iz}) Z_z + bR_{ik} + \Delta_j d g_{ij} \\ \quad = \Delta_j (1-\vartheta_f) w_{jf}\left(H - dG_{ij} - \sum_{\forall z} c_{ijf} Z_z G_{iz}\right) \\ \qquad + \left[1 - \Delta_j \vartheta_f - (1-\Delta_j)\vartheta^u_f\right] M_f \\ \quad \text{and} \quad H - \Delta_j\, dG_{ij} - \sum_{\forall z} c_{ijf} Z_z G_{iz} \geq 0 \end{array}\right]\right\}$$

In (1), for each type-f consumer, the discrete choices are combinations (i, j, k) of a zone of residence, i, a zone of employment, j, and housing type k in the zone of residence i. The continuous-choice variables of the consumer conditional on the chosen (i, j, k) are b, floor space (or housing size) in the chosen type-k housing in zone i, and the vector $Z \equiv [Z_1, \ldots, Z_{\Im'}]$, the quantities of retail goods[5] the consumer purchases from all the $\Im'$ zones by traveling there from his residence at i. If the consumer chooses unemployment, then $j = 0$ and $\Delta_0 = 0$. If the consumer chooses employment, then $j > 0$ and $\Delta_j = 1$. For employed consumers, labor supply is implied by the choices of b and $Z \equiv [Z_1, \ldots, Z_{\Im'}]$ as we shall see.

Turning now to the form of the utility function in (1) and its parameters, note first that it is Cobb–Douglas between housing and the subutility of all retail varieties. Following Anas and Xu (1999), the consumer views each location as offering a different retail variety.[6] The coefficients α_f, $\beta_f > 0(\alpha_f + \beta_f = 1)$ are the shares of disposable income spent on retail goods and on residential floor space, respectively. $1/(1 - \eta_f)$ is the elasticity of substitution between any two retail locations (or varieties). The subutility of retail varieties is C.E.S. and takes the well-known Dixit–Stiglitz (1977) form by assuming $0 < \eta_f < 1$. The consumer has an extreme taste for retail varieties in the sense that he will "travel shop" each retail location no matter how high the price of the goods offered there. Of course, pricier locations will be shopped less frequently and we will see shortly how shopping frequency attenuates with the full price of a retail good and how that full price entails travel time and cost. In the utility function in (1), the coefficients $\iota_{z|ijf} \geq 0$ are constants measuring the *inherent*

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opportunity cost for his time since he does not allocate time to work. We have extensions in which leisure and other specifications of time are treated and the unemployed value time.

[5]By *retail* we refer to *any* nonwork trip that involves the purchase of any good or service. We ignore trip chains and retail trips that do not originate from home.

[6]In the Arrow–Debreu tradition, the retail goods available at different locations are viewed as imperfect substitutes by the consumer. In the model, retail goods offered in the same zone are viewed as perfect substitutes.

attractiveness of the retail location z for consumers who have chosen (i, j, k).[7] Note that by making $\iota_{z|ijf} = 0$ any retail location can be excluded from the retail choice set. $E_{ijk|f} \in (-\infty, +\infty)$ are constant terms—common to all consumers of skill f—that measure the inherent attractiveness of the choice discrete work–residence–housing choice (i, j, k). Finally, $e_{ijk|f} \in (-\infty, +\infty)$ are *idiosyncratic utility* constants.[8] Their distribution over the population of consumers of skill f will determine a particular type of discrete-choice model.

Now let us look at the budget constraint. The prices that are exogenously given to the consumer but are endogenous in equilibrium are R_{ik}, the rent per unit of floor space of type k in zone i; w_{jf}, the hourly wage rate paid to skill-f labor employed in a zone $j > 0$, and $p_{\Re z}$, the unit prices of retail goods sold in zone z. The subscript $\Re$ denotes the retail industry. M_f is the given nonwage income of the consumer. It consists of normal investment returns from the real estate in the region (endogenous in the model) as well as income from all sources outside the regional economy (exogenous to the model). The consumer also takes as given the unit income tax rates ϑ_f and ϑ_f^u which are exogenous to the model. For an employed consumer, earned and nonwage incomes are assumed taxed at the rate ϑ_f. The tax rate for an unemployed consumer is assumed to be $\vartheta_f^u < \vartheta_f$ and applies to nonwage income only since such a consumer does not have wage income. H is the consumer's total annual time endowment and d is the exogenous number of commutes (work days) in a year. c_{ijf} are fixed and exogenous coefficients that measure the number of retail trips necessary to purchase one unit of a retail good. G_{ij} is the expected travel time over all available travel modes of one round trip from i to j for any consumer and any purpose (work or nonwork), ignoring time-of-day variations in travel times. The corresponding monetary expected travel cost is g_{ij}. As we shall see later, these travel times and costs are determined as expected values over travel modes such as highway, public transit and "other modes" and incorporate the efficient (cost minimizing) choice of routes by consumers who travel over a congested highway network. We assume that when the consumer makes his location decisions, he does not know which mode of travel, trip pattern and network routes he will choose on any given day or, equivalently, that over a year he will choose each available mode with some known probability. At the stage of problem (1) the expected travel times and monetary costs over all modes, trip patterns, and routes are taken as given conditional on each choice (i, j).

The next step is to rewrite the budget constraint so that for the employed (hence also for commuters), the delivered price paid for a unit retail good includes the value of time spent in retail trips. Note first that the above budget formulation reveals that time is assumed valued at the after-tax wage rate of the traveler. Since the unemployed do not have wages, it is assumed here that

[7]Keeping constant prices, a consumer purchases more from a location with a higher inherent attractiveness.

[8]*Idiosyncratic utility* captures the horizontal taste differentiation among consumers. We assume that idiosyncratic tastes fluctuate for the same consumer from one time period to another.

their value of time for location and retail travel choices is zero. Hence, for an unemployed traveler, the *delivered price* of a retail good purchased in zone z consists of that good's sales price at z plus just the monetary travel cost required to purchase one unit. We rearrange the budget constraint equation in (1) so that it reads:

$$(2)\quad \sum_{\forall z}[p_{\Re z}+c_{ijf}(g_{iz}+\Delta_j(1-\vartheta_f)w_{jf}G_{iz})]Z_z+bR_{ik}$$
$$=\Delta_j(1-\vartheta_f)w_{jf}(H-dG_{ij})+\left[1-\Delta_j\vartheta_f-(1-\Delta_j)\vartheta_f^u\right]M_f-\Delta_j\,dg_{ij}$$

In (2), the *full delivered price* of a retail good z for a consumer of type f residing in i and working in j is:

$$(3)\quad \psi_{z|ijf}\equiv p_{\Re z}+c_{ijf}(g_{iz}+\Delta_j(1-\vartheta_f)w_{jf}G_{iz})$$

The right side of the budget (2) is defined as the full after-tax income of the consumer net of commuting expenditures inclusive of the value of time spent commuting. More briefly, this is net full income:

$$(4)\quad \Psi_{ijf}\equiv\Delta_j(1-\vartheta_f)w_{jf}(H-dG_{ij})+\left[1-\Delta_j\vartheta_f-(1-\Delta_j)\vartheta_f^u\right]M_f-\Delta_j\,dg_{ij}$$

The word "net" expresses the fact that the full income is net of taxes as well as net of commuting costs. The word "full" expresses the fact that this income is what the consumer would have if he spent his entire time net of commuting cost, working. Actual money income from wages is lower because the consumer must take time away from potential work to make his retail trips.

For each (i, j, k), the *Marshallian demands* of the consumer which are the solution to the continuous choices in the utility maximization problem (1) are:

$$(5)\quad b_{ijk|f}=\beta_f\frac{\Psi_{ijf}}{R_{ik}}$$

$$(6)\quad Z_{z|ijf}=\frac{\iota_{z|ijf}^{\frac{1}{1-\eta_f}}\psi_{z|ijf}^{\frac{1}{\eta_f-1}}}{\sum_{\forall s}\iota_{s|ijf}^{\frac{1}{1-\eta_f}}\psi_{s|ijf}^{\frac{\eta_f}{\eta_f-1}}}\alpha_f\Psi_{ijf}$$

With these utility maximizing choices, each employed consumer also determines his annual labor supply (H_{ijf}, in hours for $j>0$) to the chosen labor market at j by subtracting his total travel time, spent on commutes and retail trips, from his annual time endowment. This is then,

$$(7)\quad H_{ijf}=H-dG_{ij}-\sum_{\forall z}c_{ijf}Z_{z|ijf}G_{iz}\geq 0$$

The next step is to evaluate the direct utility function at the maximized choices of floor space and retail quantities to get the indirect utility function which takes the form $U^*_{ijk|f}=\tilde{U}_{ijk|f}+e_{ijk|f}$ where the nonidiosyncratic part is:

$$\tilde{U}_{ijk|f} = \alpha_f \ln \alpha_f + \beta_f \ln \beta_f + \ln \Psi_{ijf} - \beta_f \ln R_{ik} + \frac{\alpha_f(1-\eta_f)}{\eta_f} \ln\left(\sum_{\forall z} \iota_{z|ijf}^{\frac{1}{1-\eta_f}} \psi_{z|ijf}^{\frac{\eta_f}{\eta_f - 1}}\right) + E_{ijk|f} \tag{8}$$

The final step requires the specification of the distribution of the idiosyncratic utilities in order to derive the probability that a randomly selected consumer will choose a discrete state (i, j, k). It is possible to derive generalized probit or nested logit models by assuming alternative correlation structures among the idiosyncratic utilities of the discrete alternatives. Such nested specifications may prove to be the best in future empirical applications. For the present we assume that for each skill group f, $e_{ijk|f} \sim$ i.i.d. Gumbel with dispersion parameter λ_f. This gives rise to the well-known multinomial logit choice probabilities:

$$P_{ijk|f} = \frac{\exp(\lambda_f \tilde{U}_{ijk|f})}{\sum_{\forall(s,t,n)} \exp(\lambda_f \tilde{U}_{stn|f})}, \quad \sum_{\forall(i,j,k)} P_{ijk|f} = 1 \tag{9}$$

Once consumers choose among the discrete states (i, j, k), including $j = 0$, they must make mode choice decisions. Suppose that there are three modes available for both commuting and retail trips ($m = 1$ (auto), $m = 2$ (transit), $m = 3$ (other)). These mode choice decisions are treated in TRAN, not in RELU. TRAN is discussed in Section 4.

Producers

Our treatment of industry structure and interindustry interactions extends that developed in Anas and Kim (1996), where industries were assumed to have Cobb–Douglas constant returns production functions with all inputs being essential in production. Here, we have generalized to a technology in which the inputs are business capital and groups of labor, buildings, and intermediate inputs from the other industries. We also include inputs of each type that may be located in other regions.[9] Within each group, individual inputs, such as intermediate inputs originating from specific zones, are treated as closely related imperfect substitutes with a constant elasticity of substitution. Across business capital and input groups, the overall production function is Cobb–Douglas and constant returns to scale so that firm size and the number of firms in an industry will be indeterminate but industry output in each model zone will be determined.

As in Anas and Kim (1996), each industry can potentially produce in every zone of the model and can import its inputs of any input group (except buildings) from all other zones. There are $r = 1, \ldots, \Re$ *basic* industries.[10] We define

[9]Business capital is perfectly elastically supplied and need not be local to the region.

[10]In the sense of Lowry (1964), basic industry means exporting industry and, in our case, includes retail trade because our retail goods can also be exported (or imported) as is true in modern cities.

the first $\Re - 1$ of these as industries producing various goods and services that are either exported from the region or are sold as intermediate inputs to other industries producing in the region. Agriculture, manufacturing and business services would be highly aggregated examples. The $\Re$th basic industry is retail trade (already encountered in Section 2 from the consumer side). Retail trade obtains intermediate inputs from the first $\Re - 1$ industries but it exports part of its output or sells it only to consumers in the region. In addition to intermediate inputs, the basic industries also use primary inputs. These are business capital, labor of each skill level, and floor space in each type of building except residential.

In addition to the basic industries, we also define two specialized industries for each building type. Thus there are $2\aleph$ such industries where $\aleph = \aleph_1 + \aleph_2$. One of these provides construction and the other demolition of that type of building. We will assume that these industries (hereafter, con–dem industries) use the primary inputs as well as inputs from potentially all basic industries and sell their services (construction and demolition) only to building developers who are described later in this section.

The production function of the rth industry ($r = 1, \ldots, \Re + 2\aleph$) with output X_{rj} in zone j is:

$$(10) \qquad X_{rj} = A_{rj} K^{\nu_r} \left(\sum_{f=0}^{F} \kappa_{f|rj} L_f^{\theta_r} \right)^{\frac{\delta_r}{\theta_r}} \left(\sum_{k=0}^{\aleph} \chi_{k|rj} B_k^{\zeta_r} \right)^{\frac{\mu_r}{\zeta_r}} \prod_{s=1}^{\Re} \left(\sum_{n=0}^{\Im'} \upsilon_{sn|rj} Y_{sn}^{\varepsilon_{sr}} \right)^{\frac{\gamma_{sr}}{\varepsilon_{sr}}}$$

This function is Cobb–Douglas and is rendered constant returns by $\nu_r + \delta_r + \mu_r + \sum_{s=1}^{\Re} \gamma_{sr} = 1$. Each of these positive coefficients will be the cost share of the respective group of inputs. For example, K is business capital with cost share ν_r. The first group of inputs, L_f, with collective cost share δ_r is labor. The firm can hire all skills of labor and the elasticity of substitution between any two skills is $1/(1 - \theta_r) > 1$. $f = 0$ stands for labor hired by the firm but located outside the region that contributes to production in this region. The second group of inputs, B_k, is buildings with cost share μ_r and elasticity of substitution $1/(1 - \mu_r) > 1$. $k = 0$ stands for buildings rented by the firm outside the region that contribute to production in this region. Then, there are also $\Re$ input groups, Y_{sn}, one for each group of intermediate inputs received from a basic industry s (including industry r itself, since a product can be used as an input in its own production). It is assumed that as an industry r uses inputs from any other industry s, it can get such inputs from potentially any zone n where industry s produces, because outputs of the same industry s produced in different model zones ($n = 0, 1, \ldots, \Im'$) are imperfectly substitutable intermediate inputs as per the Armington (1969) assumption of trade theory. The cost share of the receiving industry r for the inputs received from basic industry s is γ_{sr} and the elasticity of substitution for the sth group is $1/(1 - \varepsilon_{sr}) > 1$. $n = 0$ stands for intermediate inputs purchased and located outside the region that contribute to production. We need to comment on the input-specific constants $\kappa_{f|rj}, \chi_{k|rj}, \upsilon_{sn|rj} \geq 0$. These allow us to specify input-specific biases including the case of zero values to rule out specific inputs. For example, suppose that businesses do not use residential buildings. We would set $\chi_{k|rj} = 0$, when k stands for residential building. The leading scale factor, A_{rj}, is a constant that we vary not only by industry but also by location in order to account for place-specific Hicksian-neutral productivity effects.

Each firm solves the following cost minimization problem:

$$\text{(11)} \qquad \underset{K,[L_f],[B_k],[Y_1],\ldots,[Y_{\Re}]}{\text{Min}} \rho K + \sum_{f=0}^{F} w_{jf} L_f + \sum_{k=0}^{\aleph} R_{jk} B_k + \sum_{s=1}^{\Re} \sum_{n=0}^{\Im'} (p_{sn} + \sigma_s g_{nj}) Y_{sn}$$

subject to a target output X_{rj} given by the production function (10). p_{sn} is the price of the output sold at the place of production s and $\hat{p}_{snj} \equiv p_{sn} + \sigma_s g_{nj}$ is the delivered price of the same output purchased by other producers located at some zone j. Recall that g_{nj} is the monetary cost of commuting from n to j. σ_s is a factor that converts this passenger transport cost to the monetary cost of freight transport per unit of industry s output.[11] ρ is the exogenous price of business capital (i.e., the real interest rate) and, as we saw earlier, w_{jf} are hourly wage rates per unit of labor and R_{jk} are rents per unit of floor space. Firms are competitive in all markets and take all these prices as given.

The choice of inputs expressed as the conditional input demand functions of the industry $r = 1, \ldots, \Re + 2\aleph$ producing in zone j are as follows:

$$\text{(12)} \qquad K_{rj} = \left(\frac{1}{\rho}\right) \nu_r p_{rj} X_{rj}$$

$$\text{(13)} \qquad L_{f|rj} = \frac{\kappa_{f|rj}^{\frac{1}{1-\theta_r}} w_{jf}^{\frac{1}{\theta_r-1}}}{\sum_{z=0}^{F} \kappa_{z|rj}^{\frac{1}{1-\theta_r}} w_{jz}^{\frac{\theta_r}{\theta_r-1}}} \delta_r p_{rj} X_{rj}$$

$$\text{(14)} \qquad B_{k|rj} = \frac{\chi_{k|rj}^{\frac{1}{1-\zeta_r}} R_{jk}^{\frac{1}{\zeta_r-1}}}{\sum_{z=0}^{\aleph_2} \chi_{z|rj}^{\frac{1}{1-\zeta_r}} R_{jz}^{\frac{\zeta_r}{\zeta_r-1}}} \mu_r p_{rj} X_{rj}$$

$$\text{(15)} \qquad Y_{sn|rj} = \frac{\upsilon_{sn|rj}^{\frac{1}{1-\varepsilon_{sr}}} \hat{p}_{snj}^{\frac{1}{\varepsilon_{sr}-1}}}{\sum_{y=0}^{\Im'} \upsilon_{sy|rj}^{\frac{1}{1-\varepsilon_{sr}}} \hat{p}_{syj}^{\frac{\varepsilon_{sr}}{\varepsilon_{sr}-1}}} \gamma_{sr} p_{rj} X_{rj}$$

It is interesting to interpret the interindustry demands from the perspective of input–output analysis. The intermediate inputs flows, $Y_{sn|rj}$, from

[11]Freight cost congestion could also be modeled explicitly as related to the flow of intermediate goods between zones but this is ignored in the present.

industry s in zone n to the receiving industry r in zone j, divided by the output, X_{rj}, of the receiving industry give the *physical interindustry input–output coefficients*. These are:

$$\widehat{a}_{sn\to rj} \equiv \frac{Y_{sn|rj}}{X_{rj}} = \frac{\upsilon_{sn|rj}^{\frac{1}{1-\varepsilon_{sr}}}\hat{p}_{snj}^{\frac{1}{\varepsilon_{sr}-1}}}{\sum_{y=0}^{\Im'}\upsilon_{sy|rj}^{\frac{1}{1-\varepsilon_{sr}}}\hat{p}_{syj}^{\frac{\varepsilon_{sr}}{\varepsilon_{sr}-1}}}\gamma_{sr}p_{rj} \tag{16}$$

On the other hand, the value-based interindustry input–output coefficients, are:

$$a_{sn\to rj} \equiv \frac{\hat{p}_{snj}Y_{sn|rj}}{p_{rj}X_{rj}} = \widehat{a}_{sn\to rj}\frac{\hat{p}_{snj}}{p_{rj}} = \frac{\upsilon_{sn|rj}^{\frac{1}{1-\varepsilon_{sr}}}\hat{p}_{snj}^{\frac{\varepsilon_{sr}}{\varepsilon_{sr}-1}}}{\sum_{y=0}^{\Im'}\upsilon_{sy|rj}^{\frac{1}{1-\varepsilon_{sr}}}\hat{p}_{syj}^{\frac{\varepsilon_{sr}}{\varepsilon_{sr}-1}}}\gamma_{sr} < 1; \tag{17}$$

$$\sum_{s=1}^{\Re}\sum_{n=0}^{\Im'}a_{sn\to rj} = \sum_{s=1}^{\Re}\gamma_{sr} < 1$$

There are three striking contrasts between our general equilibrium approach and traditional input–output models. First, our assumed technology is not Leontieff but, as we saw, a Cobb–Douglas technology with C.E.S. subproduction functions. Second, our value-based interindustry coefficients are not fixed but variable: they reflect fully the effects of changing product prices in both the receiving and sending industries, so that ceteris paribus less is purchased from a far away than a close-by input location. Third, as we shall see below, both final demands for products and industry demands for primary factors are fully endogenous. In conventional input–output analysis, final demands are treated as exogenous and the demands for primary inputs are computed as gross residuals after interindustry demands are found.

Since constant returns to scale is assumed to prevail, substitution of the conditional demands (12)–(15) into the direct cost expression (11), gives the result that the industry (and any firm) makes (zero profit) at any level of output. The zero-profit condition has the constant-returns property that the price equals average or marginal cost, independent of the level of output.

$$p_{rj} = \frac{\rho^{\nu_r}}{A_{rj}\delta_r^{\delta_r}\mu_r^{\mu_r}\nu_r^{\nu_r}\left(\prod_{s=1}^{\Re}\gamma_{sr}^{\gamma_{sr}}\right)}\left(\sum_{f=0}^{F}\kappa_{f|rj}^{\frac{1}{1-\theta_r}}w_{jf}^{\frac{\theta_r}{\theta_r-1}}\right)^{\frac{\delta_r(\theta_r-1)}{\theta_r}}\left(\sum_{k=0}^{\aleph}\chi_{k|rj}^{\frac{1}{1-\zeta_r}}R_{jk}^{\frac{\zeta_r}{\zeta_r-1}}\right)^{\frac{\mu_r(\zeta_r-1)}{\zeta_r}}$$
$$\times\prod_{s=1}^{\Re}\left(\sum_{n=0}^{\Im'}\upsilon_{sn|rj}^{\frac{1}{1-\varepsilon_{sr}}}\hat{p}_{sn|rj}^{\frac{\varepsilon_{sr}}{\varepsilon_{sr}-1}}\right)^{\frac{\gamma_{sr}(\varepsilon_{sr}-1)}{\varepsilon_{sr}}} \tag{18}$$

Landlords

Our model of landlord behavior is from Anas (1982) and was used in the Anas and Arnott model (1991, 1997). It determines the short-run supply of floor space in buildings that will be kept vacant at market rents. For this purpose, residential and commercial buildings are combined into $k = 1, \ldots, \aleph$ types where $\aleph \equiv \aleph_1 + \aleph_2$. A landlord operates floor space in buildings by maximizing profit under perfect competition. The only decision a landlord takes in this model is whether to offer a unit amount of floor space for rent or withhold it from the rental market. Let $D_{iko} - d_{iko}$ and $D_{ikv} - d_{ikv}$ be the costs of maintenance that will be incurred on a unit floor space in zone i in the event that it is occupied by a tenant versus in the event that it is vacant. D_{iko} and D_{ikv} are the costs that are common to all landlords of type-k buildings in zone i and $(d_{iko}, d_{ikv}) \in (-\infty, +\infty)$ are i.i.d. Gumbel idiosyncratic costs that vary across these landlords with dispersion parameter ϕ_{ik}. Then, each landlord in each time period maximizes profit by comparing the revenue obtained under occupancy, $R_{ik} - D_{iko} + d_{iko}$, and that accruing under vacancy, $-D_{ikv} + d_{ikv}$. The decision will be resolved differently by each landlord depending on that period's draw of the idiosyncratic cost. Let q_{ik} be the probability that the landlord will decide to let a unit amount of space be occupied. From the binomial logit calculus, this probability is:

$$q_{ik}(R_{ik}) = \frac{\exp[\phi_{ik}(R_{ik} - D_{iko})]}{\exp[\phi_{ik}(R_{ik} - D_{iko})] + \exp[\phi_{ik}(-D_{ikv})]} \tag{19}$$

while $1 - q_{ik}$ is the probability that the unit floor space will remain vacant. In the very beginning of the time period, before idiosyncratic uncertainty is resolved, landlords do not yet know if they will opt for vacancy or occupancy, hence the expected rental profit from a unit amount of floor space is:

$$\omega_{ik}(R_{ik}) = \frac{1}{\phi_{ik}} \ln(\exp[\phi_{ik}(R_{ik} - D_{iko})] + \exp[\phi_{ik}(-D_{ikv})]) \tag{20}$$

which also follows from the binomial logit calculus. The rental on vacant land is taken as exogenous.

Developers

The behavior of developers is a slight simplification of that in the perfect foresight dynamic model of Anas and Arnott (1991, 1997). Since there is perfect foresight, the year-end asset prices fully reflect the future and a developer can be modeled as looking forward 1 year at a time. As explained earlier, developers buy the services of specialized (nonbasic) industries that construct or demolish buildings.[12] They are profit maximizing and competitive firms taking building asset prices, unit construction and demolition prices, and other costs as given.

[12]In Anas–Arnott, developers can also make direct conversions from one building type to another without demolition. This has been ignored in the present.

Developers determine how much of a given amount of land should remain vacant (and available for the future) or if a particular type of building should be built on it. In this way, developers determine structural density on individual lots and on the aggregate. We assume that the structural densities of the $k = 1, \ldots, \aleph$ building types are predetermined. Let m_k be the exogenous structural density (square feet of floor space per acre of lot size) of building type k. For each square foot of such floor space that is constructed, $1/m_k$ acres of land are used up. For each square foot demolished, $1/m_k$ acres of land are freed up becoming land available for future construction.

Developers face construction and demolition (industry) prices (per unit floor space) given by $p_{\Re+k,i}$ (for construction) and $p_{\Re+\aleph+k,i}$ (for demolition). In addition, developers operating in zone i face unit per acre nonfinancial costs, $\mathbb{C}_{i0k}$, for construction and unit per square foot nonfinancial costs, $\mathbb{C}_{ik0}$, for demolition of a type-k building. Define $\mathbb{C}_{i00} = 0$. Also, let ς_{i0k} and ς_{ik0} be the uncertain (fluctuating) nonfinancial idiosyncratic costs so that total nonfinancial costs are: $\mathbb{C}_{i0k} - \varsigma_{i0k}$ for construction and $\mathbb{C}_{ik0} - \varsigma_{ik0}$ for demolition. Developers are risk neutral. There is a one-period lag required for any construction (demolition) to take place. Developers buy vacant land (or a building) in the beginning of the period, then act as a landlord to operate the asset for rental during the period and decide by the end of the period on whether and what kind of building to build (or whether to demolish an existing building). This decision depends on what value of the uncertain idiosyncratic costs will be realized for a particular developer before the end of the year. Hence, when they bid for assets, developers are bidding with perfect foresight about prices but with risk neutrality about the idiosyncratic uncertainty in costs.

Let V_{ik} be the *stationary state* market price (per unit of floor space or land) of a type-k real estate asset (building) in zone i. $k = 0$ denotes vacant land as an asset and $k > 0$ stands for each of the building types in the model. The capital gains discounted to the beginning of the time period, from keeping the land undeveloped is (21a) below, from construction of a type-k building on land is (21b), from keeping a building unchanged is (21c) and from demolishing a type k is (21d), with ρ the real interest rate:

(21a) $$\Pi_{i00} = \frac{1}{1+\rho} V_{i0} + \varsigma_{i00} - V_{i0}$$

(21b) $$\Pi_{i0k} = \frac{1}{1+\rho} (V_{ik} - p_{\Re+k,i}) m_k - \mathbb{C}_{i0k} + \varsigma_{i0k} - V_{i0}$$

(21c) $$\Pi_{ikk} = \frac{1}{1+\rho} V_{ik} - \mathbb{C}_{ikk} + \varsigma_{ikk} - V_{ik}$$

(21d) $$\Pi_{ik0} = \frac{1}{1+\rho} \left(\frac{V_{i0}}{m_k} - p_{\Re+\aleph+k,i} \right) - \mathbb{C}_{ik0} + \varsigma_{ik0} - V_{ik}$$

Assume nonfinancial idiosyncratic construction costs for each i, $(\zeta_{i00}, \zeta_{i0k}; k = 1, \ldots, K) \in (-\infty, +\infty) \sim$ i.i.d. Gumbel with dispersion coefficient Φ_{i0} and also that for each (i, k) the idiosyncratic demolition costs $(\zeta_{ikk}, \zeta_{ik0}) \in (-\infty, +\infty) \sim$ i.i.d. Gumbel with dispersion coefficient Φ_{ik}. Now if a profit maximizing developer has bought a unit amount of land, he will build a type-k building on it when $\Pi_{i0k} > \Pi_{i00}$. The probability that this will occur is the $(\aleph + 1)$ nomial logit (22a). For any building type $(k > 0)$:

$$\text{(22a)} \quad Q_{iok}(V_{i0}, V_{i1}, \ldots, V_{i\aleph}) = \frac{\exp \boldsymbol{\Phi}_{i0}\left[\frac{1}{1+\rho}(V_{ik} - p_{\Re+k,i})m_k - \mathbb{C}_{i0k}\right]}{\exp\left[\frac{1}{1+\rho}\boldsymbol{\Phi}_{i0}V_{i0}\right] + \sum_{s=1,\ldots,\aleph} \exp \boldsymbol{\Phi}_{i0}\left[\frac{1}{1+\rho}(V_{is} - p_{\Re+s,i})m_s - \mathbb{C}_{i0s}\right]}$$

If the profit maximizing developer owns a type-k building, he will demolish if $\Pi_{ik0} > \Pi_{ikk}$. The probability that this will occur is the binomial logit:

(22b)

$$Q_{ik0}(V_{i0}, V_{ik}) = \frac{\exp \boldsymbol{\Phi}_{ik}\left[\frac{1}{1+\rho}\left(V_{i0}\frac{1}{m_k} - p_{\Re+\aleph+k,i}\right) - \mathbb{C}_{ik0}\right]}{\exp \boldsymbol{\Phi}_{ik}\left[\frac{1}{1+\rho}\left(V_{i0}\frac{1}{m_k} - p_{\Re+\aleph+k,i}\right) - \mathbb{C}_{ik0}\right] + \exp \boldsymbol{\Phi}_{ik}\left[\frac{1}{1+\rho}V_{ik} - \mathbb{C}_{ikk}\right]}$$

For investors in land to make zero ex ante expected economic profits (normal profits) after collecting the rent at the start of the year and paying year-end property taxes at the ad valorem tax rate τ_{i0}:

$$E[\max(\Pi_{i00}, \Pi_{i0k}; k = 1, \ldots, \aleph)] + R_{i0} - \frac{1}{1+\rho}\tau_{i0}V_{i0} = 0$$

where R_{i0} is the exogenous rent on vacant land. The expectation, $E[\bullet]$ is known from the logit calculus to be $(1/\boldsymbol{\Phi}_{i0})\ln(\text{denominator of (22a)}) - V_{i0}$. Solving the zero-profit condition for the start-of-the-year asset bid price we get the asset price for land, V_{i0}, that maintains a normal return on land. The equation gives the gross of tax asset bid price discounted to the start of the year as follows:

$$\text{(23a)} \quad \frac{1+\rho+\tau_{i0}}{1+\rho}V_{i0} = R_{i0} + \frac{1}{\boldsymbol{\Phi}_{i0}}\ln\left\{\begin{array}{l}\exp \boldsymbol{\Phi}_{i0}\frac{1}{1+\rho}V_{i0} \\ \quad + \sum_{s=1,\ldots,\aleph} \exp \boldsymbol{\Phi}_{i0}\left[\frac{1}{1+\rho}(V_{is} - p_{\Re+s,i})m_s - \mathbb{C}_{i0s}\right]\end{array}\right\}$$

For investors in type-k buildings $E[\max(\Pi_{kk}, \Pi_{k0})] + \omega_{ik}(R_{ik}) - \frac{1}{1+\rho}\tau_{ik}V_{ik} = 0$ is the zero-profit condition. $E[\bullet]$ is known from the logit calculus to be $(1/\boldsymbol{\Phi}_{ik})\ln$ (denominator of (22b)) $- V_{ik}$. Solving the zero-profit condition for the gross of tax start-of-the-year asset bid price we get:

$$\text{(23b)}\quad \frac{1+\rho+\tau_{ik}}{1+\rho}V_{ik} = \omega_{ik}(R_{ik}) + \frac{1}{\Phi_{ik}}\ln\left\{\begin{array}{l}\exp \Phi_{ik}\left(\dfrac{1}{1+\rho}V_{ik} - \mathbb{C}_{ikk}\right) \\ + \exp \Phi_{ik}\left[\dfrac{1}{1+\rho}\left(V_{i0}\dfrac{1}{m_k} - p_{\Re+\aleph+k,i}\right) - \mathbb{C}_{ik0}\right]\end{array}\right\}$$

3. GENERAL EQUILIBRIUM OF RELU

General equilibrium of the RELU model requires that five sets of conditions be satisfied conditional on **G** and **g**, the expected travel times and travel monetary costs between any zone pairs. **G** and **g** will be generated by TRAN as we shall see in Section 4:

(1) That all consumers maximize utility, all producers minimize cost and all landlords and developers maximize profits given the relevant exogenous or endogenous prices and given the expected travel times **G** and monetary travel costs **g**.

(2) Because producers are competitive and operate under constant returns to scale they earn zero economic profits: the product price covers the unit production cost (which equals average and marginal cost).

(3) That all real estate investors earn normal after-tax expected profits (zero economic profits) after competitive bidding on assets and after receiving rents and accrued capital gains and incurring construction or demolition costs.

(4) That nonwage incomes be consistent with regional building stocks, asset prices, and other sources of income from anywhere.

(5) That all markets clear with zero excess demands, requiring that there be: (i) zero excess demands in the residential and commercial markets for floor space and land in each zone of the model; (ii) that there be zero excess demand in the labor market for each skill group in each zone of the model; (iii) that the aggregate output produced by each primary industry in each zone be sufficient to meet exports and interindustry demands for the product and, in the case of retail trade, sufficient to meet consumer demands and exports.

This section presents the equations that must hold for the above conditions to be satisfied. Before we do this, it is useful to decide on some notational matters. First, we need to identify the unknown vectors that will be solved to satisfy general equilibrium. These are the vectors of product prices (**p**), wages (**w**), and rents (**R**), the vector of industry outputs (**X**), the vector of real estate asset prices (**V**), the vector of stationary real estate stocks (**S**), and the matrices of expected travel times (**G**) and expected travel monetary costs (**g**). A subscript on any vector denotes a subvector. For example, $\mathbf{w}_j \equiv [w_{j1}, \ldots, w_{jF}]$, is the wage subvector for the labor market in zone j, $\mathbf{R}_j \equiv [R_{j1}, \ldots, R_{j\aleph}]$ is the subvector of building rents in zone j. $\mathbf{p}_R \equiv [p_{\Re 1}, \ldots, p_{\Re\Im}]$ is the vector of retail prices in all model zones, while $\mathbf{p}_s \equiv [p_{s1}, \ldots, p_{s\Im}]$ is the vector of the mill prices of industry s in all model zones. $\mathbf{G} \equiv [G_{ij}]$ and $\mathbf{g} \equiv [g_{ij}]$ are the full matrices of interzonal expected travel times and monetary costs, while $\mathbf{G}_i \equiv [G_{i1}, \ldots, G_{i\Im'}]$, $\mathbf{g}_i \equiv [g_{i1}, \ldots, g_{i\Im'}]$, ${}_i\mathbf{g} \equiv [g_{1i}, \ldots, g_{\Im' i}]$ are the subvectors of round-trip expected travel times and monetary costs from zone i to all others and all others to zone i, respectively.

The following functions, already derived, are the building blocks of the general equilibrium. We show them here in shorthand notation as functions of endogenous variables and $\mathbf{G}$ and $\mathbf{g}$, suppressing other parameters and exogenous variables. These are the conditional input demands of producers for labor, buildings and intermediate goods: $L_{f|rj}(p_{rj}, X_{rj}, \mathbf{w}_j)$, $B_{k|rj}(p_{rj}, X_{rj}, \mathbf{R}_j)$, and $Y_{sn|rj}(X_{rj}, p_{rj}, \mathbf{p}_s, {}_j\mathbf{g})$, and the unit cost functions $p_{ri}(\mathbf{p}, \mathbf{w}_i, \mathbf{R}_i, {}_i\mathbf{g})$. For the consumer, the demands for floor space and retail goods are $b_{ijk|f}(R_{ik}, w_{jf}, M_f, \mathrm{G}_{ij}, g_{ij})$, $Z_{z|ijf}(\mathbf{p}_R, w_{jf}, M_f, G_{ij}, g_{ij}, \mathbf{G}_i, \mathbf{g}_i)$, while $P_{ijk|f}(\mathbf{p_R}, \mathbf{R}, \mathbf{w}, M_f, \mathbf{G}, \mathbf{g})$ are the consumer's probabilities for the discrete choice of zone of residence, workplace, and housing type. $q_{ik}(R_{ik})$ are the landlords' occupancy probabilities $Q_{i0k}(V_{i0}, V_{i1}, \ldots, V_{i\aleph})$ and $Q_{ik0}(V_{i0}, V_{ik})$ are the developers' discrete-choice probabilities of construction and demolition.

Nonwage Income

The per-consumer nonwage incomes, M_f, are specified by the following equation:

$$M_f(\mathbf{V}, \mathbf{S}) = \xi_f \frac{\Lambda + \mathbf{\Theta}}{N_f} > 0 \tag{24}$$

where $\Lambda = \sum_{\forall k} \sum_{\forall j} \frac{\rho}{1+\rho} V_{jk} S_{jk}$ is the aggregate discounted annual return from real estate in the region and Θ (if positive) is aggregate asset income from all other sources inside and outside the region that is owned by consumers in the region. If Θ is negative it would mean that on the net consumers in the region own assets that add up to less than the total after-tax return of their region's real estate assets. The coefficients $\xi_f (\sum_{f=1}^{F} \xi_f = 1)$ is the aggregate share of skill group f in the aggregate regional nonwage income and thus serve to distribute this aggregate among the skill groups. The aggregate nonwage income composition equation is then $\sum_{\forall f} N_f M_f = \Lambda + \mathbf{\Theta}$. Thus, (24) links nonwage incomes to local real estate values since these are likely major sources of nonwage income but treats other in- or outflows of nonwage income, Θ, as an exogenous constant.

Real Estate Rental Markets

For the $k = 1, \ldots, \aleph_1$ housing types and the $k = 1, \ldots, \aleph_2$ commercial building types, the demand for floor space must equal its supply in each zone $\Im$. There are $\Im\aleph_1 + \Im\aleph_2$ such equations:

$$(25)\quad \sum_{\forall f} N_f \sum_{\forall j} P_{ijk|f}(\mathbf{p}_R, \mathbf{R}, \mathbf{w}, M_f, \mathbf{G}, \mathbf{g}) b_{ijk|f}(R_{ik}, w_{jf}, M_f, G_{ij}, g_{ij}) = S_{ik} q_{ik}(R_{ik})$$

$$(26)\quad \sum_{\forall r} B_{k|ri}(p_{ri}, X_{ri}, \mathbf{R}_i) = S_{ik} q_{ik}(R_{ik})$$

where (25) are for residential floor space and (26) for business floor space.

Labor Markets

In each of the $j = 1, \ldots, \Im$ model zones, the annual demand for the labor hours of each of the $f = 1, \ldots, F$ skill groups must equal the labor hours supplied by those skill groups. Consequently, there are $F\Im$ such equations:

$$(27)\quad \sum_{r=1}^{\Re+2\aleph} L_{f|rj}(p_{rj}, X_{rj}, \mathbf{w}_j)$$

$$= N_f \sum_{\forall i,k=1}^{\aleph_1} \left[H - dG_{ij} - c_{ijf} \sum_{\forall z} G_{iz} Z_{z|ijf}(\mathbf{p}_R, w_{jf}, M_f, G_{ij}, g_{ij}, \mathbf{G}_i, \mathbf{g}_i) \right]$$

$$\times P_{ijk|f}(\mathbf{p}_R, \mathbf{R}, \mathbf{w}, M_f, \mathbf{G}, \mathbf{g})$$

Product Markets

For each basic industry, the aggregate output of that industry in a model zone j can be either directly exported from that zone or shipped to be used as an intermediate input to any other basic industry including the same industry, or to any construction/demolition industry in any model zone. Letting Ξ_{ri} be the exogenous export demands, this implies the following $\Im(\Re - 1)$ equations that clear the basic product markets:

$$(28)\quad \sum_{n=1,\ldots,\Re+2\aleph} \sum_{s=1,\ldots,\Im} Y_{ri|ns}(X_{ns}, p_{ns}, \mathbf{p}_r, {}_i\mathbf{g}) + \Xi_{ri} = X_{ri}$$

In the case of the retail industry, we assume that some output can be exported directly from the zone where the retail occurs with the rest being purchased by consumers who shop at that zone. There are $\Im$ such equations, one for each model zone:

$$(29)\quad \sum_{\forall f} N_f \sum_{\forall(n,s,k)} P_{nsk|f}(\mathbf{p}_R, \mathbf{R}, \mathbf{w}, M_f, \mathbf{G}, \mathbf{g})$$

$$\times Z_{|nsf}(\mathbf{p}_R, w_{sf}, M_f, G_{ij}, g_{ij}, \mathbf{G}_n, \mathbf{g}_n) + \Xi_{\Re i} = X_{\Re i}$$

Unlike the basic industries, the outputs of the construction and demolition industries are measured directly as the specific type of floor space constructed and demolished. Thus,

$$(30)\quad X_{\Re+s,i} = m_s S_{i0} Q_{i0s}(V_{i0}, V_{i1}, \ldots, V_{i\aleph}) \quad \text{and} \quad X_{\Re+\aleph+s,i} = S_{is} Q_{is0}(V_{is}, V_{i0})$$

are the floor-space output of each $s = 1, \ldots, \aleph$ construction industry in zone i and the floor-space output of each $s = 1, \ldots, \aleph$ demolition industry in zone i, respectively.

Normal Returns

As we saw in Section 2, all firms in the model make zero economic profit. This also means that global investors who lend financial capital to these firms make only a normal return equal to the interest rate ρ. The zero-profit equations, (18), have the following shorthand form and there are $\Im\left(\Re + 2\aleph\right)$ such equations:

(31)
$$p_{ri} - p_{ri}(\mathbf{p}, \mathbf{w}_i, \mathbf{R}_i, {}_i\mathbf{g}) = 0$$

In real estate asset markets, there are $\aleph + 1$ potential assets representing vacant land and the $\aleph$ building types potentially present in every zone. Investors in these assets make zero after-tax expected profits according to the equations derived in Section 2. There are $\Im$ such equations, (23a), for vacant land assets (one for each model zone) and each is of the form:

(32a)
$$V_{i0} - V_{i0}(V_{i0}, V_{ik}; k = 1, \ldots, \aleph) = 0$$

There are also, $\Im\aleph$ such equations for each building type, (23b), situated in each model zone:

(32b)
$$V_{ik} - V_{ik}(R_{ik}, V_{i0}, V_{ik}) = 0$$

Building Stock Conversions

In stationary equilibrium, for each type of building, the flow of demolished floor space equals that constructed so that the stock of each building type in each model zone remains stable in each year. See Anas and Arnott (1997). There are, therefore, $\Im\aleph$ such equations:

(33a)
$$S_{ik}Q_{ik0}(V_{i0}, V_{ik}) = m_k S_{i0} Q_{i0k}(V_{i0}, V_{i1}, \ldots, V_{i\aleph})$$

Meanwhile, in each model zone, the total amount of land, J_i, is given. Hence the acres taken up by each real asset type including land that remains vacant, must add up to J_i. This requires $\Im$ more equations, one for each model zone.

(33b)
$$\sum_{k=0,\ldots,\aleph} \frac{1}{m_k} S_{ik} = J_i \quad (m_0 \equiv 1)$$

Together with the $\aleph$ flow equations above, this last equation implies that, in a stationary state, the land taken up by construction in a model zone must be replenished by the land created by demolition in the same model zone. Only stationary dynamics is considered here. See Anas and Arnott (1997) for the relationship between the stationary state and the nonstationary dynamics.

TABLE 1: Equations and Unknowns in the RELU-TRAN General Equilibrium

	Equation numbers in text	Endogenous variables	Count of equations and unknowns
Floor space markets		Rents, $\mathbf{R}$	
Residential	(25)	$\mathbf{R}_1, \ldots, \mathbf{R}_{\aleph_1}$	$\Im\aleph_1 = 28$
Commercial	(26)	$\mathbf{R}_{\aleph_1+1}, \ldots, \mathbf{R}_{\aleph}$	$\Im\aleph_2 = 28$
Labor markets	(27)	Wages, $\mathbf{w}$	
By skill level		$\mathbf{w}_1, \ldots, \mathbf{w}_{\mathbf{F}}$	$\Im F$
Industry outputs		Outputs, $\mathbf{X}$	
Nonretail basic	(28)	$\mathbf{X}_1, \ldots, \mathbf{X}_{\Re-1}$	$\Im(\Re - 1) = 42$
Retail basic	(29)	$\mathbf{X}_{\Re}$	$\Im = 14$
Construction-demolition	(30)	$\mathbf{X}_{\Re+1,\ldots,\aleph_1+\aleph_2}$	$2\Im(\aleph_1 + \aleph_2) = 112$
Normal returns (production)	(18)	Product prices, $\mathbf{p}$	
Basic industries		$\mathbf{p}_{1,\ldots,\Re}$	$\Im\Re = 56$
Con–dem industries		$\mathbf{p}_{\Re+1,\ldots,\aleph_1+\aleph_2}$	$2\Im(\aleph_1 + \aleph_2) = 112$
Normal returns (development)		Asset prices, $\mathbf{V}$	
Buildings (resid., comm.)	(23a),	$\mathbf{V}_{1,\ldots,}\mathbf{V}_{\aleph_1+\aleph_2}$	$\Im(\aleph_1 + \aleph_2) = 56$
Land	(23b)	$\mathbf{V}_0$	$\Im = 14$
Stocks of real estate		Floor spaces, $\mathbf{S}$	
Buildings	(33a),	$\mathbf{S}_1, \ldots, \mathbf{S}_{\aleph_1+\aleph_2}$	$\Im(\aleph_1 + \aleph_2) = 56$
Land	(33b)	S_0	$\Im = 14$
Highway network equilibrium	(42)	**FLOW**	
		$\mathbf{R}$, $\mathbf{w}$, $\mathbf{X}$, $\mathbf{p}$,	$L = 68$
		$\mathbf{V}$, $\mathbf{S}$, **FLOW**	$\Im(7\aleph_1 + 7\aleph_2 + 2\Re + F + 2) + \mathrm{L}$
Total equations and unknowns			
$\Im = 14$, $\Re = 4$, $\aleph_1 = 2$, $\aleph_2 = 2$, $F = 4$, $\mathrm{L} = 68$			656

Summary

Given $\mathbf{G}$ and $\mathbf{g}$, the structure of expected interzonal transport travel times and monetary costs, the above completes the description of the general equilibrium system of RELU, the regional economy and land-use model. The general equilibrium comprises a total of $\Im(7\aleph + 2\Re + F + 2)$ equations in the same number of unknowns. These unknowns are: $\mathbf{R}$ (rents for each type of floor space in each model zone), $\mathbf{w}$ (wages for each skill type of labor in each model zone), $\mathbf{p}$ (prices of output for the basic and con–dem industries in each model zone), $\mathbf{X}$ (outputs of the basic and con–dem industries in each model zone), $\mathbf{V}$ (the stationary asset prices of each type of floor space and vacant land in each model zone), $\mathbf{S}$ (the stationary stocks of each type of floor space and vacant land in each model zone). Table 1 summarizes the count of equations and unknowns in the RELU general equilibrium referring to the equation numbers in the text. In the bottom of the table, the TRAN equations, described in the next section, are also included.

4. TRAN: HIGHWAY USER EQUILIBRIUM

An output of the general equilibrium of RELU is the origin to destination matrix of all *two-way daily person trips* from any origin zone i to any destination j and back to origin i, including commuting trips from residences at i to workplaces at j plus discretionary trips from residences in zone i to retail destinations in zone $j > 0$. These are found by summing over RELU functions evaluated at their equilibrium values:[13]

(34)

$$RELUTRIPS_{ij} = \left(\sum_{\forall f} N_f \sum_{k=1}^{\aleph_1} P_{ijk|f}\right) + \left(\frac{1}{d}\right)\left(\sum_{\forall f} N_f \sum_{s=0}^{\Im'} \sum_{k=1}^{\aleph_1} P_{isk|f} c_{isf} Z_{j|isf}\right); \quad j > 0, i = 1, \ldots, \Im',$$

where $P_{ijk|f} = P_{ijk|f}(\mathbf{p}_R, \mathbf{R}, \mathbf{w}, M_f, \mathbf{G}, \mathbf{g})$ and $Z_{j|isf} = Z_{j|isf}(\mathbf{p}_R\, w_{sf}, M_f, \mathrm{G}_{is}, \mathrm{g}_{is}, \mathbf{G}_i, \mathbf{g}_i)$

The role of the transportation model, TRAN, begins with *RELUTRIPS* which stands for the conventional origin to destination trip matrix of travel forecasting. Taking these from RELU, TRAN performs several transportation tasks. First, the trips are split by mode of travel into auto, transit, and other modes. Second, the trips choosing the auto mode are assigned to the road-link highway network (see Figure 1) by performing a stochastic user equilibrium that determines congested travel times for each link on the network. At present, congestion in the nonauto modes is not modeled and time of day variations in travel are ignored.

Let $\pi_{m|ij}$ be the probabilities that a trip originating in zone i and terminating in zone j will choose mode m. These mode choice probabilities come from the following mode choice model where $MODE_{ij}$ is the set of modes, which is a subset of auto ($m = 1$), transit ($m = 2$), other ($m = 3$), available to trips originating from i and going to j:

$$\text{(35)} \qquad \pi_{m|ij} = \frac{\delta_{m|ij} \exp \lambda(C_{m|ij} + \hbar_{m|ij})}{\sum_{n=1}^{3} \delta_{n|ij} \exp \lambda(C_{n|ij} + \hbar_{n|ij})}, \quad \sum_{m=1}^{3} \pi_{m|ij} = 1$$

$\delta_{m|ij} = 1$ if $m \in MODE_{ij}$ and $\delta_{m|ij} = 0$ if $m \notin MODE_{ij}$. $\delta_{m|ij} = 1$ for at least one m. $C_{m|ij} = \tilde{c}_{m|ij} + \tilde{c}_{m|ji}$ for $m \in MODE_{ij}$ are the expected round trip generalized costs of travel per person by mode m from zone i to zone j. $\lambda > 0$ is the mode choice cost-dispersion coefficient and $\hbar_{m|ij}$ are the mode-specific constants. The one-way per person generalized costs $\tilde{c}_{m|ij}$ are defined as follows for each of the modes:

$$\text{(36a)} \qquad \tilde{c}_{1|ij} = -\frac{1}{LAMDA} \ln \sum_{s \in ROUTES_{ij}} \exp(-LAMDA \times RGCOST_{s \in ROUTES_{ij}})$$

[13]To avoid duplicating RELU notation, some TRAN variables and parameters are defined in capital letter "words."

(36b) $\tilde{c}_{2|ij} = (VOT)t_{ij2} + \nu_{ij2}, \quad \text{if} \quad \delta_{2|ij} = 1$

(36c) $\tilde{c}_{3|ij} = (VOT)t_{ij3} + \nu_{ij3}, \quad \text{if} \quad \delta_{3|ij} = 1$

(36a), for the auto mode, is the expected minimized generalized cost of travel over all the routes in the set of routes, $ROUTES_{ij}$, connecting the origin zone i and the destination zone j. It comes from the route-choice model which will be discussed below. $LAMDA > 0$ is the route-choice model's cost-dispersion coefficient. $RGCOST$, the generalized cost of travel by auto, obtained from the route-choice model. In (36b), for transit and (36c) for the "other" mode, VOT is *the value of time for travel on the network,* assumed uniform across all travelers. t_{ijm}, $m \geq 2$ is the travel time from i to j for a nonauto mode, and ν_{ijm}, $m \geq 2$, is the nonauto traveler's monetary cost for travel from i to j. Since congestion in nonauto modes is not modeled, these nonauto travel times and monetary costs are treated as constants.

Now, turning to the auto mode, given that $\tilde{\eta}$ is a constant car occupancy rate in persons per vehicle, then the *one-way vehicle trips* originating at zone $i = 1, \ldots, \Im'$ and terminating at zone $j > 0$ that must be allocated to the highway network are:

$$(37) \qquad TRIPS_{ij} = [RELUTRIPS_{ij} \times \pi_{1|ij} + RELUTRIPS_{ji} \times \pi_{1|ji}] \left(\frac{1}{\tilde{\eta}}\right)$$

These one-way auto vehicle trips for each origin–destination zone pair are assigned to a highway network with a link-node representation (see Figure 1). The nodes correspond exactly to RELU zone centroids and the links are crude aggregations of the underlying road system. A congested highway network equilibrium is then calculated based on the stochastic cost minimization model of network equilibrium of Daganzo and Sheffi (1977), solved according to the fixed point formulation of Anas and Kim (1990).

For the route-choice model, define the following notation. Let $i = 1, \ldots, \Im'$ be zone centroids of trip origins (residential locations) and $j = 1, \ldots, \Im'$ zone centroids of trip destinations (work or retail locations). Let $\ell = 1, \ldots, L$ be the links on the aggregated highway network. Let $ROUTES_{ij}$ be the *set* of permissible highway routes, each route being itself a *set* of sequential links on the highway network, connecting origin i with destination j. $r \in ROUTES_{ij}$ is such a highway route that belongs to the above set. $a_{\ell \in r \in ROUTES_{ij}}$ is zero or one depending on link ℓ belonging (or not) to route $r \in ROUTES_{ij}$. Let $PROB_{r \in ROUTES_{ij}}$ be the probability that an *auto vehicle trip* originating at node i and terminating at node j will choose route $r \in ROUTES_{ij}$. This probability is given by the following multinomial logit, assuming that the travelers choose the route that minimizes their perceived generalized cost of travel. See, for example, Daganzo and Sheffi (1977):

$$(38) \qquad PROB_{r \in ROUTES_{ij}} = \frac{\exp\left(-LAMDA \times RGCOST_{r \in ROUTES_{ij}}\right)}{\sum\limits_{\forall s \in ROUTES_{ij}} \exp\left(-LAMDA \times RGCOST_{s \in ROUTES_{ij}}\right)}$$

As noted earlier, $LAMDA > 0$ is the route cost-dispersion coefficient and $RGCOST_{r\in ROUTES_{ij}}$ is the route's generalized cost of travel per vehicle calculated from:

$$(39) \qquad RGCOST_{r\in ROUTES_{ij}} = \sum_{\forall \ell} a_{\ell\in r\in ROUTES_{ij}} \times \left[VOT \times TIME_{\ell} + MCOST_{\ell}\right]$$

The earlier equation (36a) gives the expected minimized cost over the set $ROUTES_{ij}$. $TIME_{\ell}$, the congested one-way travel time (in minutes) on link ℓ is calculated from the following congestion function:

$$(40) \qquad TIME_{\ell} = ALPHA_{\ell} \times LENGTH_{\ell}\left(1 + BETA_{\ell}\left[\frac{FLOW_{\ell}}{CAP_{\ell}}\right]^{CEXP}\right)$$

where $LENGTH_{\ell}$ is the length of the highway link in miles, $APLHA_{\ell}$ is the inverse vehicle free-flow (uncongested) speed on link ℓ in minutes per mile, $BETA_{\ell}$ and $CEXP$ are coefficients and CAP_{ℓ} is the capacity of link ℓ. $FLOW_{\ell}$ is the *vehicle* traffic flow on a highway link to be determined endogenously in the highway network equilibrium. $MCOST_{\ell}$ is the one-way *per traveler* monetary cost on a link and is assumed not to depend on congestion. It is calculated from,

$$(41) \qquad MCOST_{\ell} = UCOST_{\ell} \times LENGTH_{\ell} \times \left(\frac{1}{\tilde{\eta}}\right)$$

where $UCOST_{\ell}$ is the exogenous unit monetary cost in \$ per v*ehicle*-mile. The generalized cost of travel on a particular route on the network, $RGCOST$, and the expected minimized cost are then calculated by plugging in (40) and (41) into (39) and then (39) into (36a).

In TRAN, the congested equilibrium network flows on each highway link are calculated from the Anas–Kim (1990) simultaneous equations at the link level,

$$(42) \qquad FLOW_{\ell} = \sum_{\forall ij} TRIPS_{ij} \sum_{r\in ROUTES_{ij}} PROB_{r\in ROUTES_{ij}} \times a_{\ell\in r\in ROUTES_{ij}}, \quad \forall \ell$$

after substituting (40) and (41) into (39), the resulting (39) into (38), and the resulting (38) into (42). Then, given $TRIPS_{ij}$ precalculated from (37), the resulting L equations in (42) are solved simultaneously for the uniquely determined fixed point equilibrium $FLOW_{\ell}$, $\ell = 1, \ldots, L$.

What remains now is to use the output of TRAN to calculate the travel times and costs, $\mathbf{G} \equiv [G_{ij}]$ and $\mathbf{g} \equiv [g_{ij}]$ used by consumers to make choices in RELU:

$$(43) \qquad G_{ij} = \sum_{m=1}^{3} \pi_{m|ij}(t_{ijm} + t_{jim})$$

$$(44) \qquad g_{ij} = \sum_{m=1}^{3} \pi_{m|ij}(\nu_{ijm} + \nu_{jim})$$

where t_{ijm}, v_{ijm} for $m \geq 2$ are the zone-to-zone times and costs for the nonauto modes. For the auto mode, these times and costs are made consistent with the congested highway equilibrium by:

$$(45) \qquad t_{ij1} = \sum_{r \in ROUTES_{ij}} PROB_{r \in ROUTES_{ij}} \left(\sum_{\forall \ell} a_{\ell \in r \in ROUTES_{ij}} TIME_{\ell} \right)$$

and

$$(46) \qquad v_{ij1} = \sum_{r \in ROUTES_{ij}} PROB_{r \in ROUTES_{ij}} \left(\sum_{\forall \ell} a_{\ell \in r \in ROUTES_{ij}} MCOST_{\ell} \right)$$

5. TESTING THE UNIFIED RELU-TRAN© ALGORITHM

The unification of RELU and TRAN requires that the **G** and **g** derived by solving TRAN (Section 4) are in fact those **G** and **g** that give rise to the *RELUTRIPS* matrix and all other quantities produced by solving RELU (Section 3). Separate algorithms have been designed and coded to compute the RELU stationary general equilibrium and the TRAN stochastic user equilibrium to high degrees of numerical precision. The joint equilibration (unification) of RELU and TRAN is achieved by linking these two algorithms together so that the joint algorithm converges to a high degree of numerical precision. Figure 2 shows how the two algorithms are linked together to unify the models. Figures 3 and 4 are flowcharts of the RELU and TRAN algorithms, respectively.

Results of testing the RELU-TRAN algorithm are reported for the 14 zone (plus one peripheral zone) delineation of the Chicago MSA according to Figure 1. The road network shown in the figure is a somewhat crude aggregation of the underlying Chicago road network. Each aggregated road connecting a zone pair represents two links, one going in one direction and the other going in reverse. RELU-TRAN has been calibrated for the above 14 + 1 zone, 68-link setting using data circa 1990. The four basic RELU industries, $r = 1, \ldots, 4$ are: (i) agriculture; (ii) manufacturing; (iii) business services; (iv) retail. The four building types, $k = 1, \ldots, 4$ are: (i) single family residential; (ii) multiple family housing; (iii) industrial; (iv) commercial. The populations in the four skill groups, $f = 1, \ldots, 4$ correspond to the personal income quartiles in 1990. The calibrated model produces an equilibrium that simultaneously and exactly satisfies all of the equilibrium conditions of RELU-TRAN, which we have described in Sections 2–4.

As shown in Table 1, this Chicago version of the RELU model consists of 588 equations linked to each other in a special block recursive manner, while TRAN consists of 68 equations. As shown in Figure 3, the RELU algorithm is structured in a way that exploits the block-recursive nature of the equations. To solve the RELU model, one must start with exogenously given expected travel time and monetary cost matrices **G** and **g.** In the RELU-TRAN unified

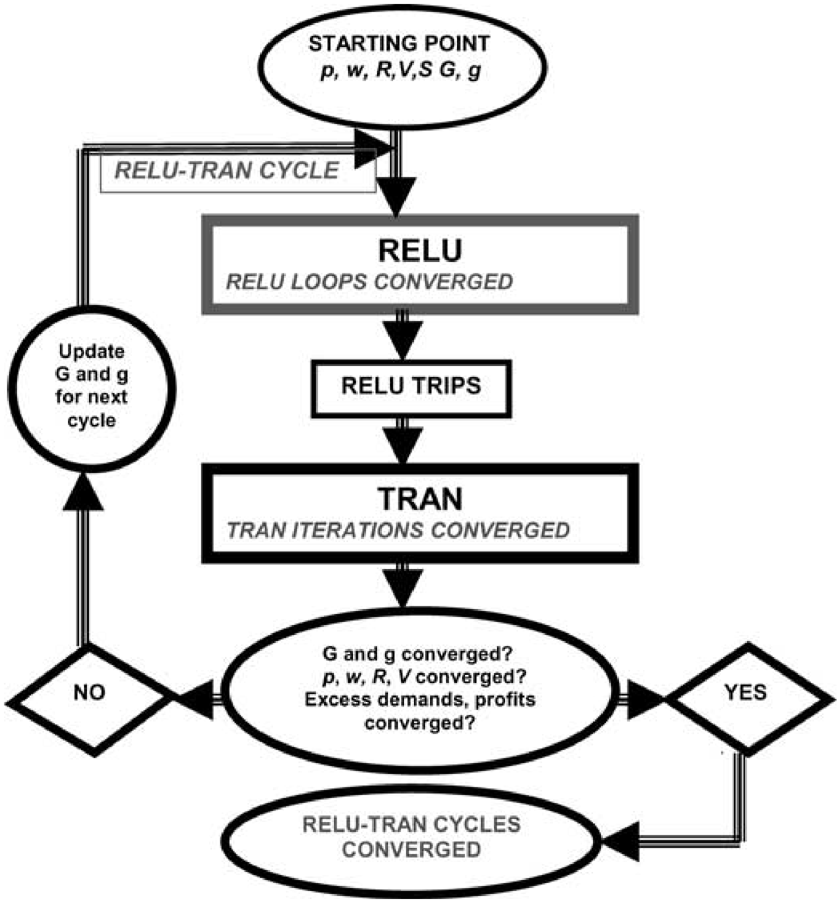

FIGURE 2: Cyclical Linking of the RELU and TRAN Algorithms in RELU-TRAN.

equilibrium, the updated values of these variables become mutually consistent with those produced by TRAN. The RELU algorithm is as follows:

STEP 0 (*Initialization*). Arbitrary exogenous vectors of rents, **R**, wages, **w**, product-prices, **p**, real estate prices, **V**, and building stocks, **S**, serve as initial guesses. Given such guesses, as well as the given **G** and **g** and all exogenous variables and parameters, the following sequentially arranged steps complete a single loop of RELU as shown in Figure 3.

STEP 1 (*Product prices*). The zero economic profit equations, (18), are solved for the vector of equilibrium prices, **p** as a fixed point, given **R**, **w**, and **g**.

STEP 2 (*Industry outputs*). Given the **p** from STEP 1, and the **S**, **R**, **w**, and **g**, equations (28), (29), and (30) are solved in the manner of a conventional

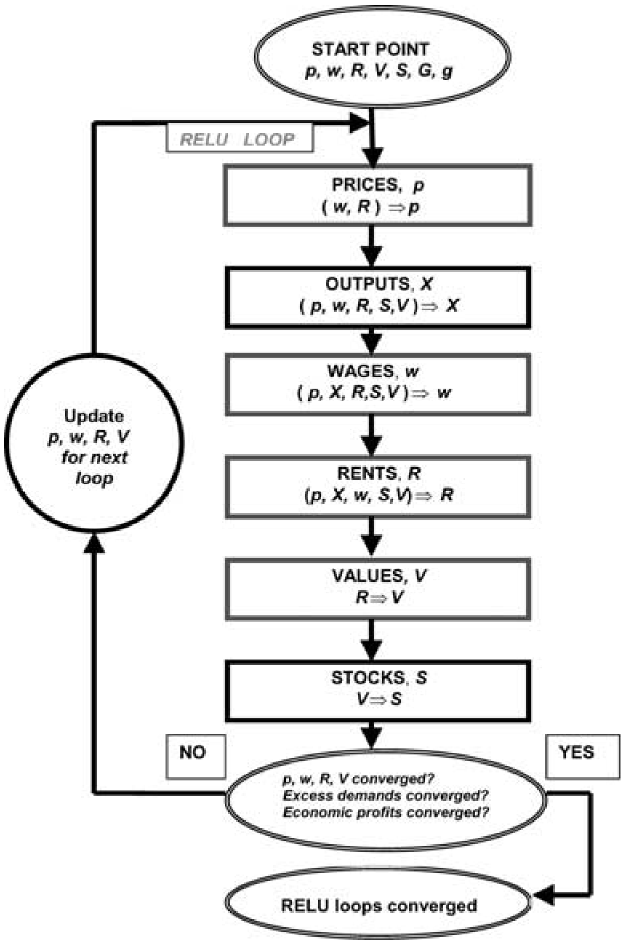

FIGURE 3: The RELU Algorithm.

input–output model to determine industry outputs, **X**. More precisely, using **g** and **p**, the delivered price vectors are calculated first and then, $(\Re + 2\aleph) \times (\Re + 2\aleph)$ value-based input–output coefficients, $a_{sn \to rj}$, for all industries in all zones, including any that are zeroes, are evaluated from (17). Equations (30) are then used to directly calculate the con–dem industry outputs and the left sides of Equations (29) are used to directly calculate the retail demands. Retail outputs by zone are then found from the left side of (29) by adding the exogenous exports to the retail demands. The value-based input–output coefficients of the nonretail $\Re - 1$ basic industries are then arranged in a matrix $\mathcal{A} = [a_{sn \to rj}]$. The classical input–output solution is then $\mathbf{X} = (\mathbf{I} - \mathcal{A})^{-1}\,\mathbf{F}$, where the right sides are the total final demands, **F**, comprising exogenous exports plus demands from the con–dem and the retail industries:

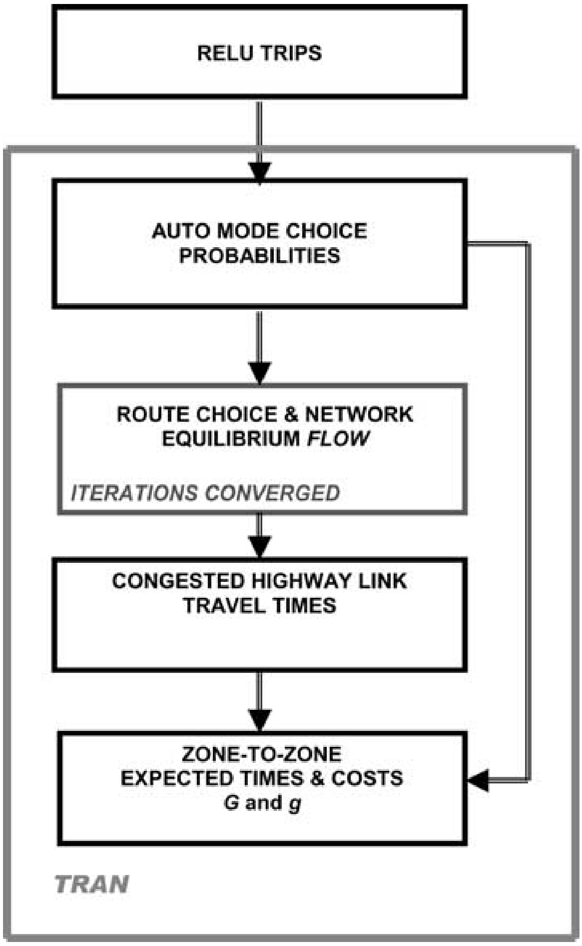

FIGURE 4: The TRAN Algorithm.

$$\begin{bmatrix} X_{11} \\ \bullet \\ \bullet \\ X_{\Re-1\Im} \end{bmatrix} = \begin{bmatrix} 1 - a_{11\to 11} & -a_{11\to 12} & \bullet & -a_{11\to \Re-1\Im} \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ -a_{\Re-1\Im\to 11} & -a_{\Re-1\Im\to 12} & \bullet & 1 - a_{\Re-1\Im\to\Re-1\Im} \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} \Xi_{11} + \left(\sum_{n=\Re+1}^{2\aleph} \sum_{s=1}^{\Im} a_{11\to ns} X_{ns} \right) + \left(\sum_{s=1}^{\Im} a_{11\to \Re s} X_{\Re s} \right) \\ \bullet \\ \bullet \\ \Xi_{\Re-1\Im} + \left(\sum_{n=\Re+1}^{2\aleph} \sum_{s=1}^{\Im} a_{\Re-1\Im\to ns} X_{ns} \right) + \left(\sum_{s=1}^{\Im} a_{\Re-1\Im\to \Re s} X_{\Re s} \right) \end{bmatrix}$$

STEP 3a (*Commercial Rents*). Given **p** from STEP 1 and **X** from STEP 2, and given **S,** the $\Im\aleph_2$ commercial floor space excess demand equations, (26) are solved simultaneously to determine an update of the commercial part of the rent vector, **R**.

STEP 3b (*Residential Rents*). Given **S** and **V**, the nonwage incomes, **M**, are calculated from (24) and given **p** from STEP 1, and also **w**, **G**, and **g**, the residential floor space excess demand equations (25) are solved simultaneously to determine an updated residential rent vector, **R**.

STEP 4 (*Wages*). Given the rent vector, **R**, from STEP 3a and STEP 3b, the price vector, **p**, from STEP 1, the outputs **X** from STEP 2 and the nonwage incomes, **M**, calculated in STEP 3b, the labor market excess demand equations (27) are solved simultaneously to determine an updated wage vector, **w**.

STEP 5 (*Real Estate Asset Prices*). Given the rents, **R**, from STEP 3a, 3b the asset bid-price zero-profit equations (23a) and (23b) are solved simultaneously to determine the stationary real estate price vector, **V**.

STEP 6 (*Building stocks*). Given the asset prices, **V**, Equations (33a) and (33b) are solved simultaneously (separately for each model zone), to determine the stationary building stocks, **S**. This relies on knowing that once the real estate prices are found from STEP 5, the stationary construction–demolition probabilities Q_{i0k}, Q_{ik0}, $k = 0, 1, \ldots, \aleph$ are explicitly calculated and the stocks can then be found from (33a) and (33b). To see this, note that the equations can be arranged in matrix form to directly solve, as in Anas and Arnott (1997), for the vector of stationary building and land stocks in each zone i:

$$\begin{bmatrix} S_{i0} \\ S_{i1} \\ S_{i2} \\ . \\ S_{i\aleph} \end{bmatrix} = \begin{bmatrix} m_1 Q_{i01} & -Q_{i10} & 0 & . & 0 \\ m_2 Q_{i02} & 0 & -Q_{i20} & . & 0 \\ . & . & . & . & . \\ m_\aleph Q_{i0\aleph} & 0 & 0 & . & -Q_{i\aleph 0} \\ 1 & 1/m_1 & 1/m_2 & . & 1/m_\aleph \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ J_i \end{bmatrix}$$

STEP 7 (*Updating*). Combining the results of STEP 1–STEP 6, the algorithm has determined updated vectors **p**, **X**, **R**, **w**, **V**, **S** conditional on the **G** and **g** and all exogenous variables and parameters. It is checked whether these updated vectors are converged and whether they simultaneously satisfy all of the equilibrium conditions of RELU to a desired level of accuracy in the manner discussed below. If not, then the next RELU loop is started by returning to STEP 1 with these updated vectors replacing the previous values of these vectors (see Figure 3).

Iterations, Loops, Cycles. In each STEP 1, 3a, 3b, 4, and 5 and in TRAN a Newton–Raphson procedure iterates until some iteration t, when the elements of the super vector $\mathbf{x} = [\mathbf{p}, \mathbf{w}, \mathbf{R}, \mathbf{V}, \mathbf{FLOW}]$ relevant to that STEP, satisfy the maximum relative error condition and the zero-profit or excess demands satisfy

$\max_{\forall i}(|\frac{x_{i,t} - x_{i,t-1}}{(1/2)(x_{i,t} + x_{i,t-1})}|) < ITERTOL$ and $\max_{\forall i}(|\frac{LS_{i,t} - RS_{i,t}}{(1/2)(LS_{i,t} + RS_{i,t})}|) < ITERTOL$, respectively. *LS* and *RS* are left and right sides of the *i*th equation and *ITERTOL* is an arbitrarily small tolerance. With both sets of inequalities holding together, the step is considered converged. If a variable or function is still unconverged at any STEP after *MAXIT*, a maximum number of iterations, is reached at that step, then a new RELU loop begins as shown in Figure 3.

RELU is considered converged in the *t*th loop when, for all of the RELU variables being calculated, $\max_{\forall i}(|\frac{x_{i,t} - x_{i,t-1}}{(1/2)(x_{i,t} + x_{i,t-1})}|) < LOOPTOL$, where *LOOPTOL* is the RELU loop tolerance. As shown in Figure 2, RELU loops are nested within RELU-TRAN cycles. In each new RELU-TRAN cycle, after the sequence of RELU loops converge, TRAN iteratively calculates a congested highway network equilibrium as a fixed point in link flows to a desired level of accuracy by satisfying the same iteration criterion as in the RELU iterations, and then TRAN finds the new **FLOW** and calculates the new **G** and **g** returning them to RELU. See Figure 4. A new RELU-TRAN cycle then starts with these updated **G** and **g** as shown in Figure 2. RELU-TRAN cycles are considered converged in some *t*th cycle, when the **G** and **g** satisfy $\max_{\forall ij}(|\frac{G_{ij,t} - G_{ij,t-1}}{(1/2)(G_{ij,t} + G_{ij,t-1})}|, |\frac{g_{ij,t} - g_{ij,t-1}}{(1/2)(g_{ij,t} + g_{ij,t-1})}|) < CYCLETOL$ where *CYCLETOL* is the specified cycle tolerance. This procedure of Newton–Raphson iterations nested within RELU loops and RELU loops nested within RELU-TRAN cycles is continued until a convergent RELU-TRAN cycle is found and all variables and, at the end, also all excess demand and excess profits satisfy the specified tolerances.

The CPU time within which convergence is achieved depends on how a sequence of iterations–loops–cycles is induced by the choice of the tolerances and on how small the tolerances are specified to be. In general, the algorithm can be operated in what we call the *slow-cycle* or the *fast-cycle* modes. Slow-cycle mode refers to the requirement that RELU loops are converged with a small *LOOPTOL* before the algorithm is allowed to go to TRAN, while a fast-cycle mode means that a larger *LOOPTOL* is chosen so that the algorithm passes from RELU to TRAN after fewer RELU loops. Similarly, RELU loops can be operated as slow (fast) by setting *ITERTOL* so that each of the iterative steps in the RELU algorithm shown in Figure 3, can be converged to a very high (or low) degree of accuracy before the next RELU loop begins.

Performance also depends on the manner in which updated variables are passed from one iteration to the next inside each STEP of the RELU algorithm. Note that each Newton–Raphson iteration begins with some $Current(x_t)$, which is the current value of variable x, adjusted in iteration t. After adjustment, the iteration finds a new candidate value for the variable, which we will call $New(x_t)$. In the next iteration, we set $Current(x_{t+1}) = \varpi(t)New(x_t) + [1 - \varpi(t)]Current(x_t)$, where $0 \leq \varpi(t) < 1$. A sequence of weights $\varpi(1), \varpi(2), \ldots, \varpi(t), \varpi(t+1), \ldots$ starting close to zero and moving to 1 incrementally, helps avoiding divergence. For example, in all the runs reported in Table 2, $\varpi(t) = at$ and $t \leq MAXIT$ where *MAXIT* is the maximum number of iterations allowed, with

MAXIT = 3 and $a = 0.10$. We have also experimented with other methods for smoothing values of variables between iterations and with higher *MAXIT* values. The lower the *MAXIT* value, the faster is each RELU loop (fast loop) but more loops are required (slow cycle).

Our strategy for testing the RELU-TRAN algorithm was as follows. The model was calibrated in such a way that a general equilibrium solution of RELU-TRAN was known exactly through the calibration.[14] Given such a calibrated equilibrium point, we systematically started the algorithm from various large and small perturbations away from the calibrated equilibrium and observed two things:

1. *Convergence*: Whether the algorithm converged to an equilibrium point to a high degree of accuracy as specified by *MAXIT* the three tolerances (*ITERTOL, LOOPTOL,* and *CYCLETOL*) and the procedure of cycles, loops, iterations described above.
2. *Uniqueness*: Whether the equilibrium to which the algorithm converged was, to a very high degree of accuracy, the calibrated one or whether the algorithm converged to some other equilibrium.

We experimented with several smoothing and tolerance setting procedures until the algorithm converged well in all the tests. The algorithm always converged to the calibrated equilibrium for all starting points so that all the relative errors comparing converged and calibrated values were within at least 0.0001 of their calibrated equilibrium values. This strongly suggests that the RELU-TRAN system of equations have a unique equilibrium solution at least within the large neighborhood of the calibrated equilibrium from which starting points were selected.[15] We measure the performance of a convergent algorithm by several criteria. First is the CPU time required to achieve a convergent solution of the 656 equations on our machine. A second criterion is the *accuracy* of the convergence as specified by the three tolerances. A third criterion is the *monotonicity* of the algorithm's convergence path, characterized by a nonincreasing number of RELU loops within successive RELU-TRAN cycles. In all of the Table 2 runs, *ITERTOL* = *LOOPTOL* = *CYCLETOL*, set at 1×10^{-5}, unless otherwise indicated, and $a = 0.10$ and *MAXIT* = 3 as noted earlier.

There are runs, A, B, and C in Table 2, grouped according to how the calibrated equilibrium was perturbed to obtain a starting point for the first RELU-TRAN cycle. In the A group, all elements of the starting vectors **w**, **R**, **G**, **g**, and **FLOW** are perturbed by the same percentage relative to the calibrated equilibrium. For example, 120 means that all of these vectors are uniformly set at exactly 120 percent of their calibrated equilibrium values. In the B group of runs, different vectors are perturbed by different percentages, while in the

[14]How the model is calibrated will be presented in a separate paper.

[15]We have not attempted to prove uniqueness, and it remains a valid concern in future research.

TABLE 2: Testing the Convergence of the RELU-TRAN Algorithm

	Start point as percent deviation from equilibrium point				Convergence to equilibrium ($MAXIT = 3$, $a = 0.10$)			
Var	Wages **W** 56 (14 × 4)	Rents **R** 56 (14 × 4)	Time/Cost **G, g** 225 (15 × 15)	FLOW 68	TOLERANCES (*ITERTOL, LOOPTOL, CYCLETOL*)	CPU Time (Min:Sec)	RELU-TRAN cycles	RELU loops by cycle
A	***Uniform deviations***							
1	80	80	80	80	1×10^{-5}	3:53.7	8	69,54,40,27,12,6,1,1
2	90	90	90	90	1×10^{-5}	3:36.0	8	65,50,36,22,9,3,1,1
3	99	99	99	99	1×10^{-5}	2:32.0	7	51,38,18,6,2,1,1
4	101	101	101	101	1×10^{-5}	2:29.7	7	52,35,17,6,1,1,1
5	110	110	110	110	1×10^{-5}	3:37.0	8	67,53,36,20,8,3,1,1
6a	120	120	120	120	1×10^{-3}	1:46.8	5	42,28,7,2,1
6b	120	120	120	120	1×10^{-5}	3:55.6	8	69,56,37,29,11,8,2,1
6c	120	120	120	120	1×10^{-9}	11:26.2	14	128,113,94,84,75,65,54,44,32,19,7,3,1,1
6d	130	130	130	130	1×10^{-5}	4:14.5	9	72,56,41,30,15,7,1,1,1
6e	150	150	150	150	1×10^{-5}	4:37.1	9	82,63,45,35,22,7,3,1,1
B	***Nonuniform deviations***							
7	120	120	120	80	1×10^{-5}	3:56.0	8	69,56,37,29,11,8,2,1
8	120	120	80	120	1×10^{-5}	3:54.3	8	70,54,40,27,12,6,1,1
9	120	80	80	120	1×10^{-5}	3:47.4	8	61,54,40,27,12,6,1,1
10	80	80	120	120	1×10^{-5}	3:55.9	8	71,54,37,29,8,2,1,1
11	120	80	120	120	1×10^{-5}	3:50.5	8	59,56,37,29,8,2,1,1
12a	80	120	120	120	1×10^{-3}	1:42.7	5	39,23,7,2,1
12b	80	120	120	120	1×10^{-5}	3:55.1	8	68,54,37,29,11,8,2,1
12c	80	120	120	120	1×10^{-7}	7:15.1	11	98,85,68,58,46,38,23,9,3,2,1
12d	80	120	120	120	1×10^{-9}	11:22.7	14	125,113,94,84,73,65,54,44,32,19,7,3,1,1
C	***Randomized deviations***							
13a	80–120	80–120	80–120	80–120	1×10^{-3}	1:47.4	5	42,27,8,2,1
13b	80–120	80–120	80–120	80–120	1×10^{-5}	2:49.7	7	40,40,36,18,5,2,1
13c	80–120	80–120	80–120	80–120	1×10^{-5}	3:06.1	8	43,46,34,18,6,2,1,1
13d	80–120	80–120	80–120	80–120	1×10^{-9}	11:14.1	14	127,110,93,85,73,63,51,42,32,17,7,2,1,1
14	70–130	70–130	70–130	70–130	1×10^{-5}	2:56.9	8	28,37,37,23,9,3,1,1
15	60–140	60–140	60–140	60–140	1×10^{-5}	3:00.8	8	30,34,37,26,12,4,1,1
16	50–150	50–150	50–150	50–150	1×10^{-5}	3:11.7	8	52,50,30,16,8,2,1,1

C group of runs a random number generator is used to sample the value of each element of each vector within an indicated percentage range around the calibrated equilibrium. For example, in run 16 a range of 50–150 means that the starting point of each element of each vector is sampled from within 50 percent to 150 percent of the calibrated equilibrium according to the uniform distribution of percentages in that range.

In Table 2, virtually all runs exhibit essentially monotonic convergence. A slightly nonmonotonic path is encountered only in runs 13c, 14, and 15. The behavior of the algorithm exhibits the regularities that one would intuitively expect from the way the starting points and the tolerances are set.

Regularity 1 (*Starting point's "distance" from equilibrium*). In the A group of runs, increasing the deviations from equilibrium from ±1 percent to ±20 percent, while keeping the tolerances and the smoothing procedure constant, increases the number of RELU-TRAN cycles from seven to only eight but increases considerably the number of RELU loops within the corresponding cycles of the different runs.

Regularity 2 (*Random sampling of starting point*). In the C group of runs, keeping tolerances and deviations from the calibrated equilibrium constant, the different outcomes of random sampling within the same range does not materially affect the time to convergence as seen from runs 13b and 13c, for example.

Regularity 3 (*Accuracy requirement*). Comparing runs 6a–6c from group A or 12a–12d from group B, or 13a–13d from group C shows that when the deviation from the calibrated equilibrium is kept the same but the accuracy requirement is increased by decreasing the tolerance from 1×10^{-3} to 1×10^{-5} to 1×10^{-7} down to 1×10^{-9}, then the cycles required to converge, increase from 5 to 8 to 14. The loops within the corresponding cycles of the different runs also increase significantly.

The algorithm is coded in Fortran and all of the tests of the algorithm were performed on a UNIX-based Sun Microsystems Ultra Sparc server running Solaris 2.8, with CPU speed of 360 MHz and 1 gigabyte RAM. The integrity of the source code is independent of the machine environment.

6. CONCLUDING REMARKS

We have reported on the microeconomic structure of the RELU-TRAN model, the design of an algorithm for solving the model's stationary general equilibrium and the numerical testing of the algorithm for a 656-equation version of the model. As explained in the Introduction, the model is a synthesis of previous less general models by one of the authors (see References), all of which are based on discrete-choice theoretic treatments of the underlying microeconomic theory. This approach to modeling metropolitan areas is realistic, useful, and flexible and avoids some of the theoretical, empirical, and computational limitations of other approaches.

A separate paper will report on how RELU-TRAN was calibrated for the Chicago MSA and how the calibration aspects of discrete-choice-based models offer advantages over less flexible approaches. Yet other papers are being planned on the applications of the model to evaluate the social costs and benefits of a variety of urban policies. In the past, antecedents of the current model that were not as general were used by Anas and Duann (1985) to evaluate the effects of planned transportation investments on real estate prices. After the investments took place and price changes were measured, McDonald and Osuji (1995) and McMillen and McDonald (2004) claimed that the forecasts were accurate. In another application by Anas and Arnott (1997), a dynamic housing market model was used to compare conflicting aspects of taxation and of subsidies on the Chicago housing market. In Anas and Arnott (1993) the same model was used in four MSAs to compare the welfare effects of demand-side and supply-side housing subsidization policies on the welfare of consumer groups. Recently, Mansur et al. (2002) adopted the stationary version of the same dynamic model to examine the causes of homelessness in California and the effects of various policies on the homeless. Because of the general equilibrium nature of the current RELU-TRAN model, a much broader menu of policies can be evaluated than has been possible in the past. The effects of metropolitan policies spanning transportation, land use, real estate development, and industrial location and production can be evaluated simultaneously. One area of such policies concerns the impact of highway capacity expansion or new highways on economic productivity and on urban sprawl in metropolitan areas. What types of transportation investments increase aggregate metropolitan products by industry? Is a higher level of production associated with a higher degree of urban sprawl? If so, are there land-use restrictions ("smart growth") that limit sprawl while enhancing or not excessively reducing urban production? Under the current project, funded by the EPA, we are looking at such policies while the model will be expanded to incorporate energy consumption by residential and industrial activity and by automobiles, and to calculate emissions from stationary and mobile sources.

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APPENDIX: NOTATIONAL GLOSSARY

We provide a list of the RELU variables, distinguishing between dimensional indices, endogenous variables, exogenous variables, parameters for calibration, idiosyncratic variables, and intermediate variables, which are functions of parameters, idiosyncratic variables, and exogenous, endogenous variables.

Indices for Dimensions of Model

$\Im$: number of model zones spanning the region where consumers can locate and businesses produce.

$\wp$: number of peripheral zones where consumers can locate and businesses produce, treated as exogenous in the equilibrium.

$\Im' = \Im + \wp$: all zones where consumers can locate and businesses produce.

$\aleph$: number of building types.

$\aleph_1$: number of residential building types.

$\aleph_2$: number of commercial building types.

$\Re$: number of basic industries in the model.

F: number of skill levels in the labor market.

Equilibrating (Endogenous) Variables (Unknowns of the Model)

X_{ri}: aggregate industry r output produced in zone i.

p_{ri}: price of industry r output produced in zone i.

w_{if}: hourly wage rate paid to skill-f labor hired by producers in zone i.

R_{ik}: rental price of floor space in a type-k building in zone i.

V_{ik}: asset price of floor space in a type-k building in zone i.

S_{ik}: aggregate stock of floor space in a type-k building in zone i.

Exogenous Variables

J_i: aggregate land in model zone i.

N_f: number of consumers in skill level f.

H: annual hours available to each consumer for allocation between work and travel

d: commuting days per year.

Θ: aggregate income originating from outside the region but accruing to residents within the region.

ξ_f: share of total aggregate income $(\Lambda + \Theta)$ accruing to the region that is owned by skill group f.

A_{ri}: scale parameter of production function of industry r producing in model zone i.

Ξ_{ri}: export demand for product of industry r produced in model zone i.

ρ: interest rate or the opportunity cost of capital in the business sector including developers.

m_k: structural density in units of floor space per unit of land in a type-k building (=1 for vacant land).

D_{iks}: maintenance cost of a type-k unit floor space in zone i depending on vacancy $(s = v)$ or occupancy $(s = o)$ status.

$\mathbb{C}_{ik0}$: cost of demolishing a unit amount of type-k floor space in a building in model zone i.

$\mathbb{C}_{i0k}$: cost of constructing a unit amount of floor space in a type-k building in model zone i.

$\mathbb{C}_{i00}$: cost of keeping vacant a unit amount of land in model zone i.

$\mathbb{C}_{ikk}$: cost of keeping as is a unit amount of floor space in a type-k building in model zone i.

τ_{ik}: ad valorem property tax per unit of type-k floor space in model zone i paid at the end of a year by an investor.

ϑ_f: constant marginal income tax rate levied on the wage and nonwage income of a skill-f consumer who is employed.

ϑ_f^u: constant marginal income tax rate levied on the nonwage income of a skill-f consumer who is unemployed.

Consumers

Parameters

α_f: the elasticity of utility with respect to the subutility of retail goods.

β_f: the elasticity of utility with respect to housing floor space.

η_f: $\frac{1}{1-\eta_f}$ is the elasticity of substitution between any two retail varieties.

$\iota_{z|ijf}$: inherent attractiveness of retail location at model zone z for a consumer of skill f residing in model zone i and working in model zone j.

c_{ijf}: trips required by consumer of skill f residing in model zone i and working in model zone j to purchase a unit quantity of retail goods.

$E_{ijk|f}$: inherent attractiveness of residence–workplace–housing type location bundle (i, j, k) for consumer of skill f.

λ_f: dispersion parameter of i.i.d. Gumbel idiosyncratic utilities in skill group f.

Idiosyncratic Variables

$e_{ijk|f}$: idiosyncratic utility of residence–workplace–housing type location bundle (i, j, k) for consumer of skill f.

Intermediate Variables

$U_{ijk|f}$: direct utility of residence–workplace–housing type location bundle (i, j, k) for consumer of skill f.

$\tilde{U}_{ijk|f}$: common part of the indirect utility of residence–workplace–housing type location bundle (i, j, k) for consumer of skill f.

Ψ_{ijf}: full income of consumer in skill level f net of the opportunity cost of commuting and taxes.

M_f: nonwage annual income per consumer of skill level f.

Λ: aggregate annual nonwage income from assets within the region after subtracting property taxes.

$\psi_{z|ijf}$: delivered price (inclusive of the opportunity cost of travel) of retail goods produced in zone z for consumers of skill level f working in model zone j and residing in model zone i.

$Z_{z|ijf}$: Marshallian demand for retail goods in model zone z for consumer of skill level f working in model zone j and residing in model zone i.

$b_{ijk|f}$: Marshallian demand for floor space in a type-k residential building by a worker working in model zone j and residing in model zone i.

H_{ijf}: annual labor hours supplied to producers in model zone j by skill level-f consumers who reside in zone i.

$P_{ijk|f}$: probability that a skill-f consumer will choose to reside in a type-k residential building in model zone i while working in model zone j or remaining unemployed ($j = 0$).

G_{ij}: round trip travel time per person-trip from zone i to zone $j(>0)$ by a traveler of skill f.

g_{ij}: round trip monetary cost per person-trip from zone i to zone $j(>0)$ by a traveler of skill f.

$\pi_{m|ij}$: probability that a consumer residing in zone i makes a round trip to zone $j(>0)$ using mode m.

Firms

Parameters

ν_r: elasticity of output with respect to business capital used in industry r.

δ_r: elasticity of output with respect to the group of labor inputs used in industry r.

μ_r: elasticity of output with respect to the group of floor space inputs used in industry r.

γ_{sr}: elasticity of output with respect to the group of intermediate inputs from industry s used in industry r.

θ_r: $\frac{1}{1-\theta_r}$ is the constant elasticity of substitution among labor inputs used in industry r.

ζ_r: $\frac{1}{1-\zeta_r}$ is the constant elasticity of substitution among floor space inputs used in industry r.

ε_{sr}: $\frac{1}{1-\varepsilon_{sr}}$ is the constant elasticity of substitution among intermediate inputs from industry s used in industry r.

$\kappa_{f|rj}$: inherent attractiveness of labor input of skill f used by industry r in zone j.

$\chi_{k|rj}$: inherent attractiveness of floor space of type k used by industry r in zone j.

$\upsilon_{sn|rj}$: inherent attractiveness of intermediate inputs originating from industry s in zone n and used by industry r in zone j.

σ_r: coefficient for converting the monetary cost of personal transport to the monetary cost of transporting a unit amount of the output of industry r.

Intermediate Variables

$\hat{p}_{rjn}$: the delivered price of a unit output of industry r produced in model zone j and sold as an intermediate input to other producers in model zone n.

K_{rj}: business capital used as an input in production by industry r in zone j.

$L_{f|rj}$: labor of skill group f used as an input in production by industry r in zone j.

$B_{k|rj}$: floor space of type k used as an input in production by industry r in zone j.

$Y_{sn|rj}$: output of industry s produced in zone n used as an intermediate input in production by industry r in zone j.

Developers and Landlords

Parameters

ϕ_{ik}: dispersion parameter of i.i.d. Gumbel distributed idiosyncratic maintenance costs for a unit type-k floor space in model zone i.

Φ_{i0}: dispersion parameter of i.i.d. Gumbel distributed idiosyncratic costs associated with keeping land vacant or constructing on it.

Φ_{ik}: dispersion parameter of i.i.d. Gumbel distributed idiosyncratic costs associated with keeping a building as is or demolishing it.

Idiosyncratic Variables

d_{iks}: idiosyncratic maintenance cost of a type-k unit floor space in zone i depending on vacancy ($s = v$) or occupancy ($s = o$) status.

ς_{ik0}: idiosyncratic cost of demolishing a unit amount of type-k floor space in a building in model zone i.

ς_{i0k}: idiosyncratic cost of constructing a unit amount of floor space in a type-k building in model zone i.

ς_{i00}: idiosyncratic cost of keeping vacant a unit amount of land in model zone i.

ς_{ikk}: idiosyncratic cost of keeping as is a unit amount of floor space in a type-k building in model zone i.

Intermediate Variables

ω_{ik}: expected annual profit of a landlord operating a type-k unit floor space in model zone i before the idiosyncratic vacancy costs are revealed.

q_{ik}: probability that a landlord operating a type-k unit floor space in model zone i will decide to offer it for rent.

Π_{ik0}: profit of demolishing a unit amount of type-k floor space in a building in model zone i.

Π_{i0k}: profit of constructing a unit amount of floor space in a type-k building in model zone i.

Π_{i00}: profit of keeping vacant a unit amount of land in model zone i.

Π_{ikk}: profit of keeping as is a unit amount of floor space in a type-k building in model zone i.

Q_{i0k}: probability that a developer in zone i will construct a type-k building on a unit vacant land.

Q_{ik0}: probability that a developer in zone i will demolish a unit floor space in a type-k building.